Simple Harmonic Motion

Physics 1425 Lecture 28

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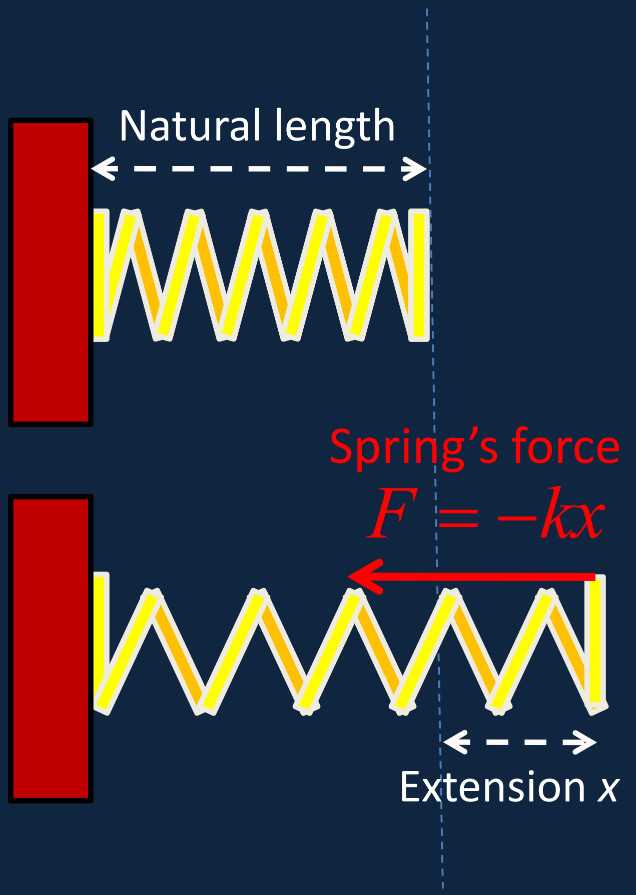
Force of a Stretched Spring

- If a spring is pulled to extend beyond its natural length by a distance *x*, it will pull back with a force

$$F = -kx$$

where *k* is called the "spring constant".

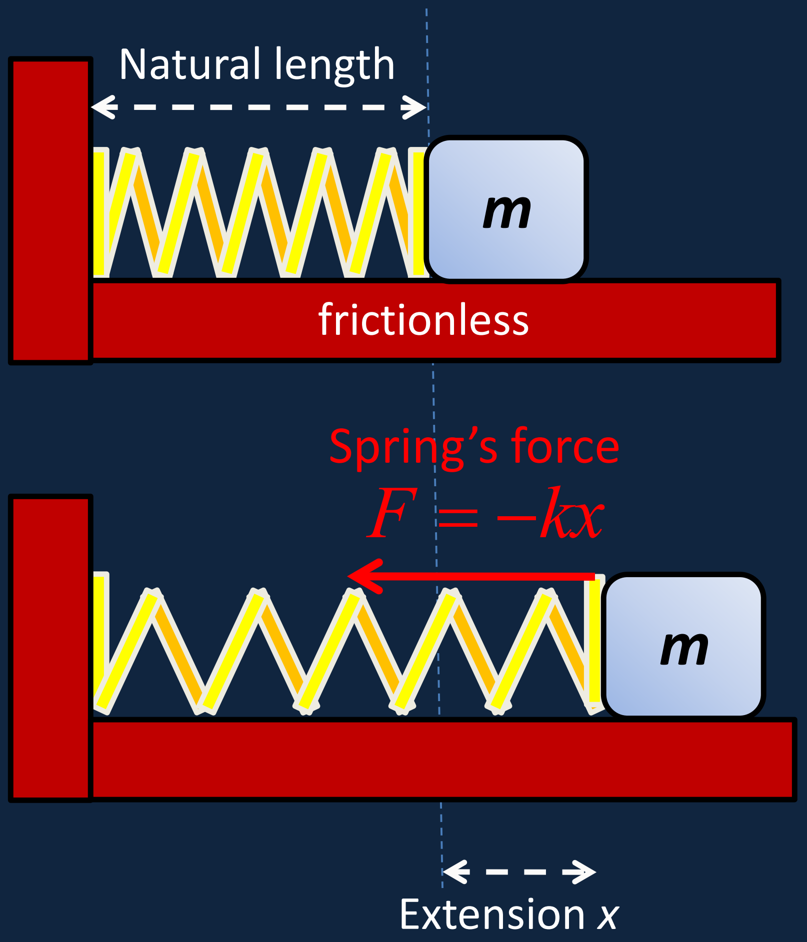
The same linear force is also generated when the spring is *compressed*.



Mass on a Spring

- Suppose we attach a mass *m* to the spring, free to slide backwards and forwards on the frictionless surface, then pull it out to *x* and let go.
- *F* = *ma* is:

$$m d^2 x / dt^2 = -kx$$



Solving the Equation of Motion

- For a mass oscillating on the end of a spring,

$$md^2x/dt^2 = -kx$$

- The most general solution is

$$x = A\cos(\omega t + \phi)$$

- Here *A* is the amplitude, ϕ is the phase, and by putting this *x* in the equation, $m\omega^2 = k$, or

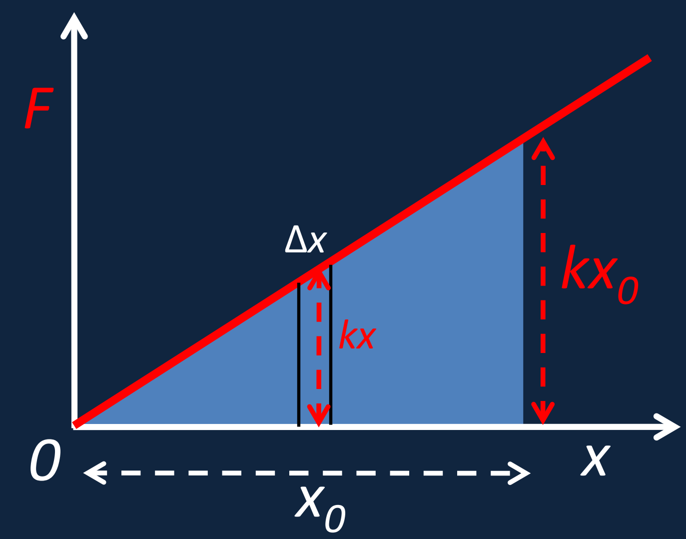
$$\omega = \sqrt{k/m}$$

- Just as for circular motion, the time for a complete cycle

$$T = 1/f = 2\pi/\omega = 2\pi\sqrt{m/k} \quad (f \text{ in Hz.})$$

Energy in SHM: Potential Energy Stored in the Spring

- Plotting a graph of external force $F = kx$ as a function of x, the work to stretch the spring from x to $x + \Delta x$ is force x distance
- $\Delta W = kx\Delta x$, so the total work to stretch the spring to x_0 is



$$W = \int_0^{x_0} kx\,dx = \tfrac{1}{2}kx_0^2$$

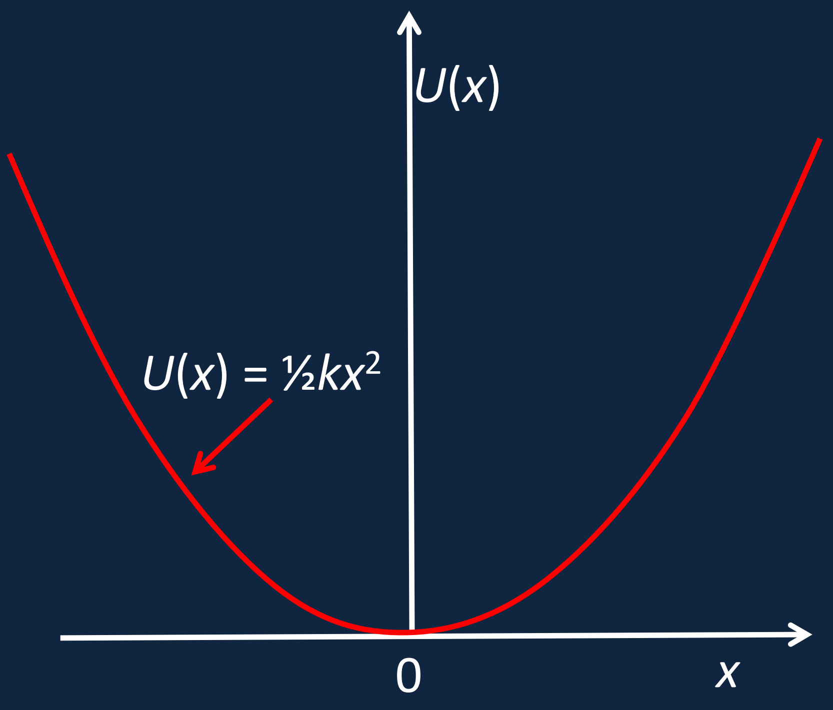
This work is stored in the spring as potential energy.

Potential Energy *U*(*x*) Stored in Spring

- The potential energy curve is a parabola, its steepness determined by the spring constant *k*.
- For a mass *m* oscillating on the spring, with displacement

$$x = A\cos(\omega t + \phi)$$

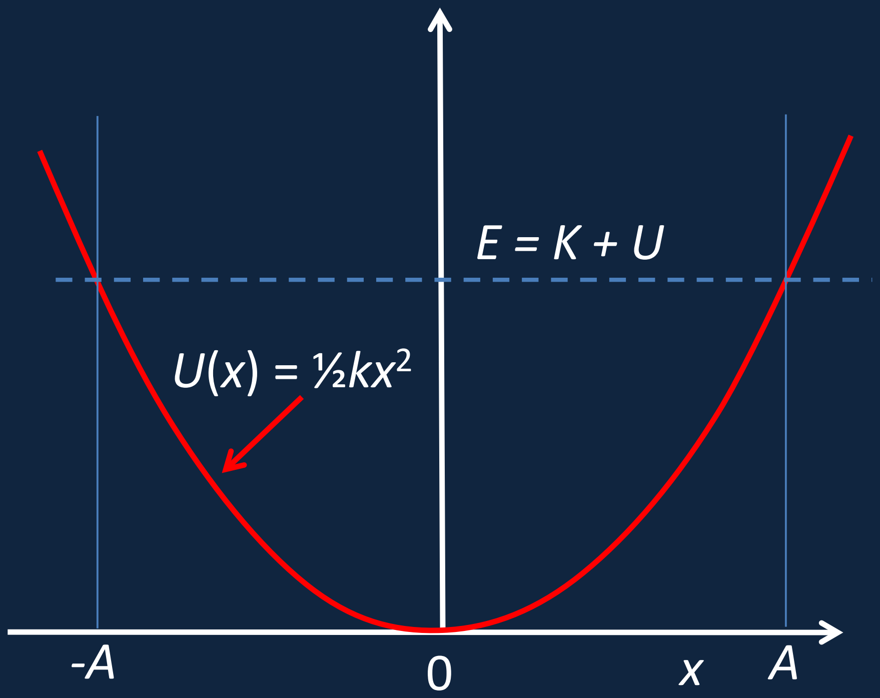
the potential energy is $U(x) = \frac{1}{2}kA^2\cos^2(\omega t + \phi)$



Total Energy *E* for a SHO

- The total energy E of a mass m oscillating on a spring having constant k is the sum of the mass's kinetic energy and the spring's potential energy:
- $E = \frac{1}{2}mv^2 + \frac{1}{2}kx^2$
- For a given E, the mass will oscillate between the points $x = A$ and $-A$, where
- $E = \frac{1}{2}kA^2$
- Maximum speed is at $x = 0$, where $U(x) = 0$, and

  $E = \frac{1}{2}mv^2$ at $x = 0$



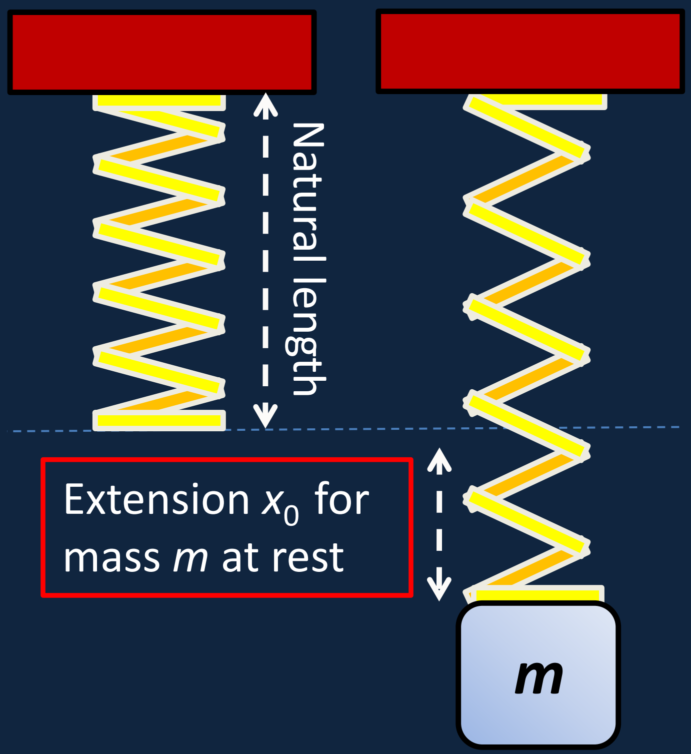
Mass *Hanging* on a Spring

- Suppose as before the spring constant is k.
- There will be an extension x_0, $kx_0 = mg$, when the mass is at rest.
- The equation of motion is now:

$$md^2x/dt^2 = -k(x - x_0)$$

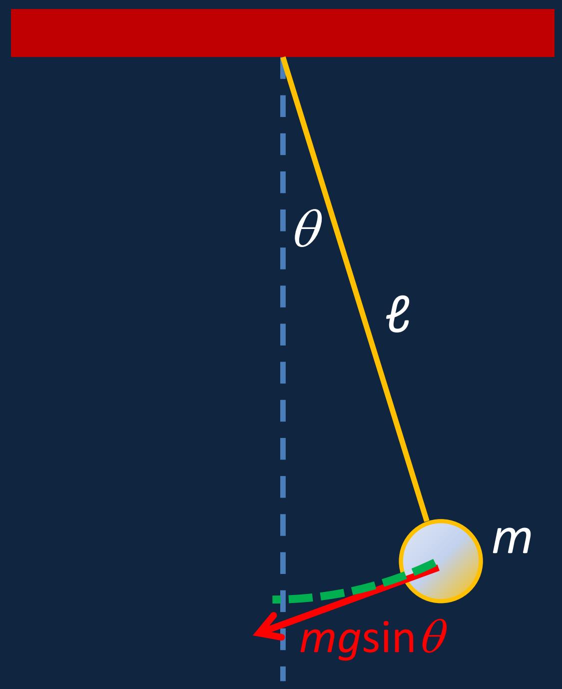
- with solution

$$x - x_0 = A\cos(\omega t + \phi), \quad \omega^2 = k/m.$$



The Simple Pendulum

- A simple pendulum has a bob, a mass m treated as a point mass, at the end of a light string of length ℓ.
- We consider only small amplitude oscillations, and measure the displacement $x = \ell\theta$ along the circular arc.
- The restoring force is $F = -mg\sin\theta \cong -mg\theta$ along the arc.



$F = ma$ for the Simple Pendulum

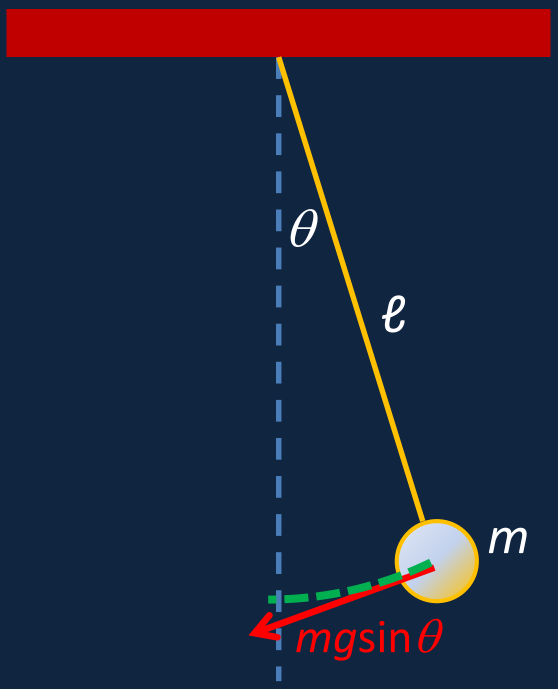
- The displacement along the circular arc is $x = \ell\theta$.
- The restoring force is

  $F = -mg\sin\theta \cong -mg\theta = -mgx/\ell$

  along the arc.
- $F = ma$ is

  $$d^2x/dt^2 = -gx/\ell$$

  (canceling out *m* from both sides!).



Period of the Simple Pendulum

- The equation of motion

$$d^2x/dt^2 = -gx/\ell$$

has solution

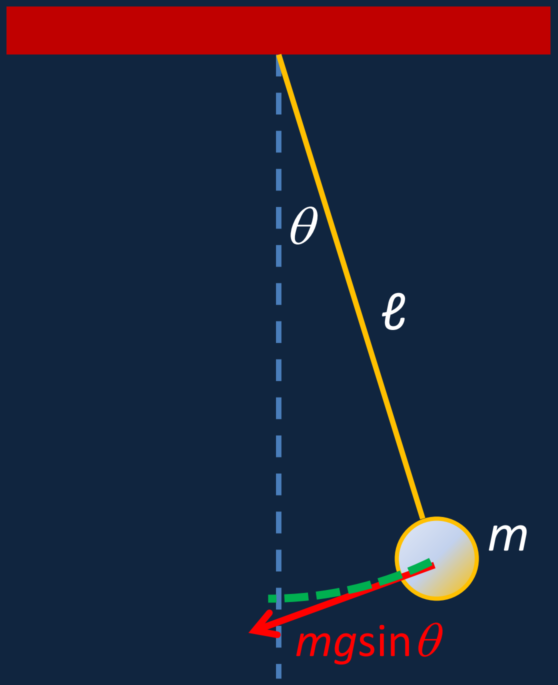
$$x = A\cos(\omega t + \phi)$$

- Here

$$\omega = \sqrt{g/\ell}$$

and the time for a complete swing

$$T = 2\pi/\omega = 2\pi\sqrt{\ell/g}.$$



The time for a complete swing doesn't depend on the mass *m*, for the same reason that different masses fall at the same rate.

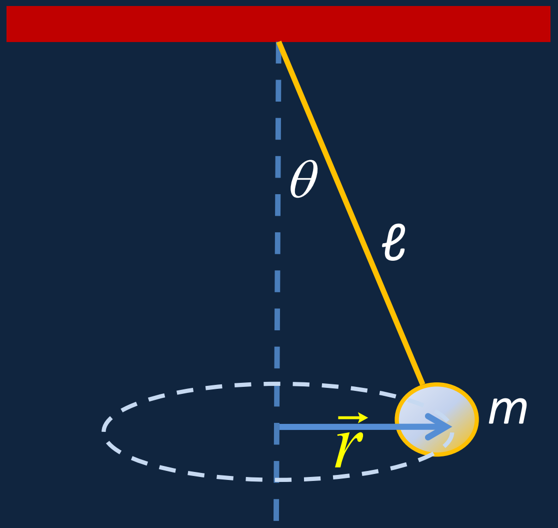
Reminder: the Conical Pendulum

- Imagine a conical pendulum in steady circular motion with small angle θ.
- As viewed from above, it moves in a circle, the centripetal force being $-(mg/\ell)\vec{r}$.
- So the equation of motion is

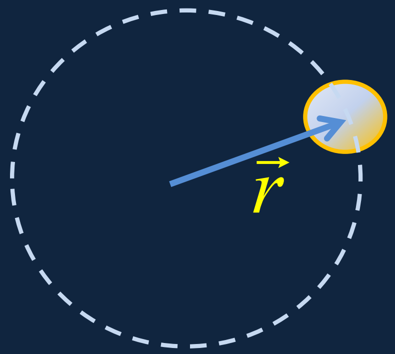
$$d^2\vec{r}/dt^2 = -(g/\ell)\vec{r}$$

and for the *x*-component of $\vec{r}$

$$d^2x/dt^2 = -gx/\ell$$



Top View:



The SHO and Circular Motion

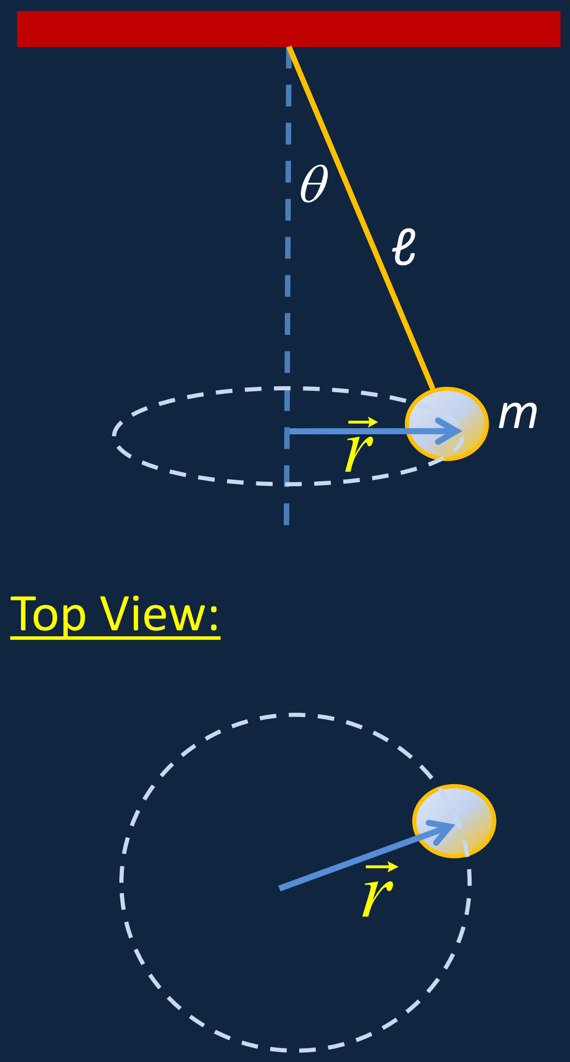
- We can now see that the equation of motion of the simple pendulum at small angles—which is a simple harmonic oscillator

$$d^2x/dt^2 = -gx/\ell$$

is nothing but the *x*-component of the steady circular motion of the conical pendulum

$$d^2\vec{r}/dt^2 = -(g/\ell)\vec{r}$$

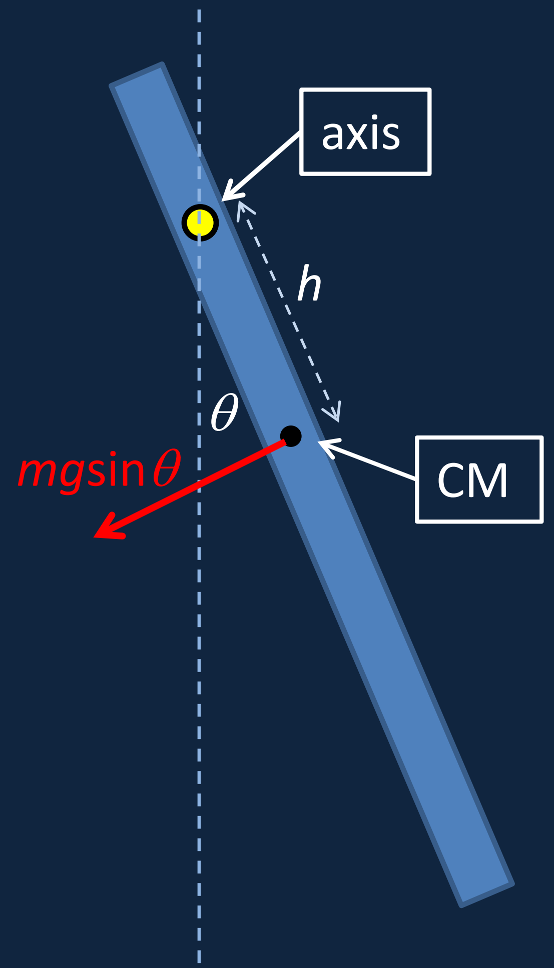
- The simple pendulum is the *shadow* of the conical pendulum, and click here to see it!



The Physical Pendulum

- The term "physical pendulum" is used to denote a rigid body free to rotate about a fixed axis, making small angular oscillations under gravity.
- Taking the distance of the CM from the axis to be *h*, at (small) angle displacement θ, the torque is

$$\tau = mgh\sin\theta \cong mgh\theta$$



$\tau = I\alpha$ for the Physical Pendulum

- In the small angle approximation, the equation of motion $\tau = I\alpha$ is

$$I\frac{d^2\theta}{dt^2} = -mgh\theta$$

- with solution

$$\theta = \theta_0 \cos(\omega t + \phi)$$

- and

$$T = 2\pi / \omega = 2\pi\sqrt{I / mgh}.$$

- Remember this is $I_{axis} = I_{CM} + mh^2$!