Damped and Driven Harmonic Motion

Physics 1425 Lecture 29

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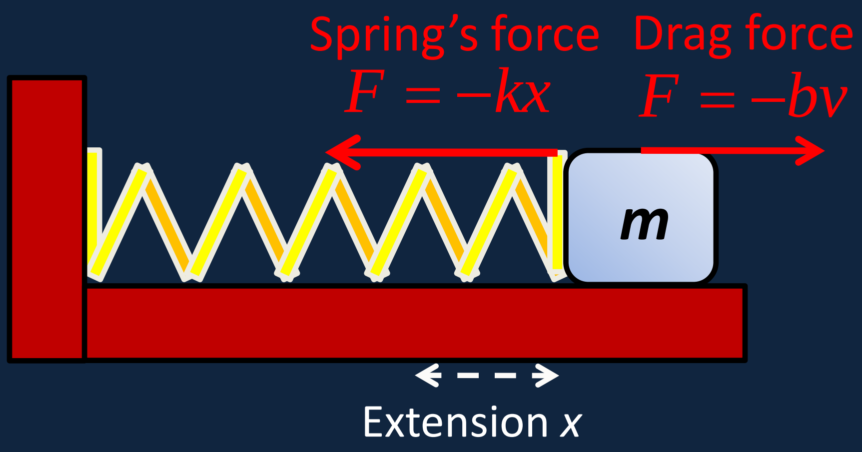
Damped Harmonic Motion

- In the real world, oscillators experience damping forces: friction, air resistance, etc.
- These forces always oppose the motion: as an example, we consider a force $F = -bv$ proportional to velocity.
- Then $F = ma$ becomes:

$$ma = -kx - bv$$

- That is, $md^2x/dt^2 + bdx/dt + kx = 0$



The direction of drag force shown is on the assumption that the mass is moving to the *left*.

Underdamped Motion

- The equation of motion

$$md^2x/dt^2 + bdx/dt + kx = 0$$

has solution

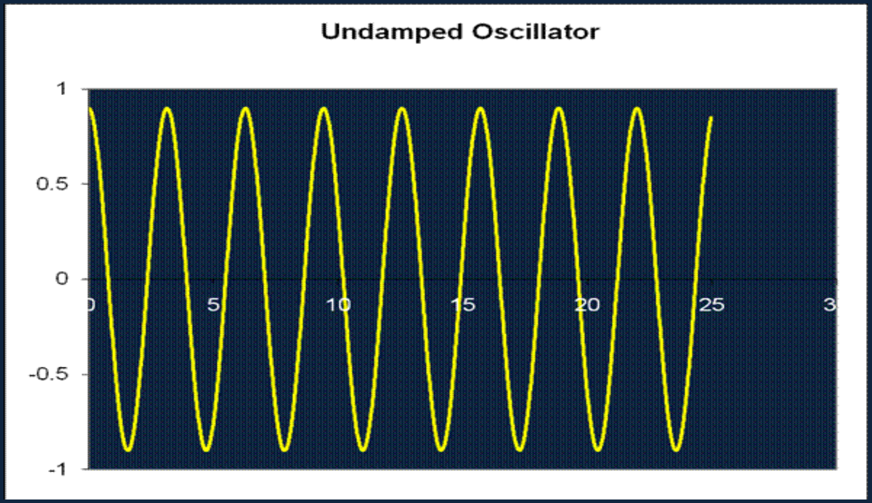
$$x = Ae^{-\gamma t}\cos\omega' t$$

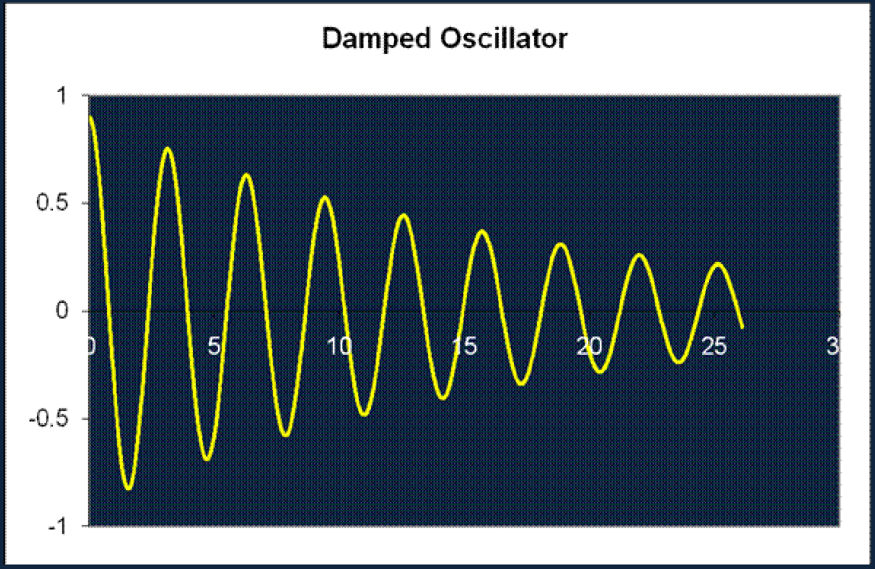
where

$$\gamma = b/2m,$$

$$\omega' = \sqrt{(k/m) - (b^2/4m^2)}$$

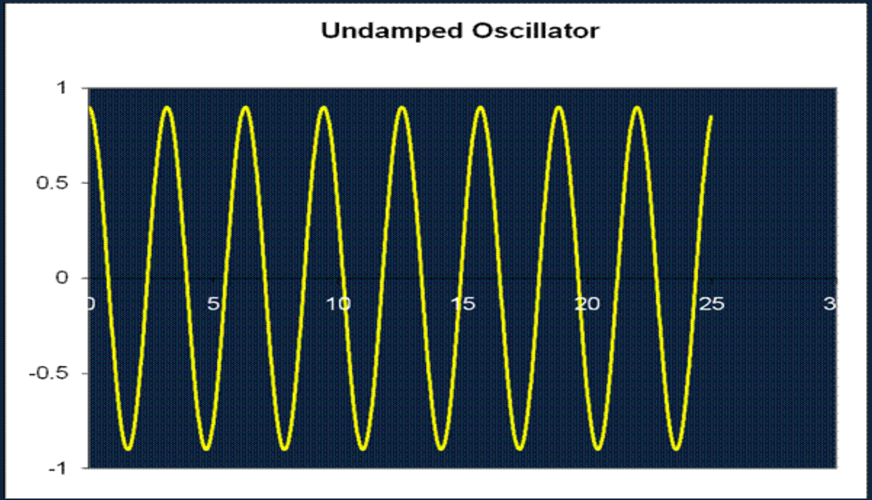
Plot: *m* = 1, *k* = 4, *b* = 0.11

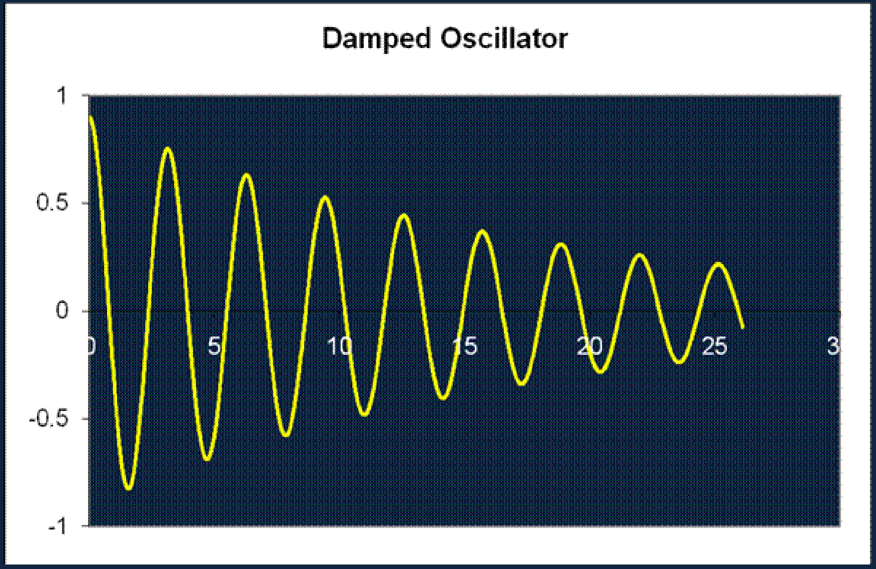




Underdamped Motion

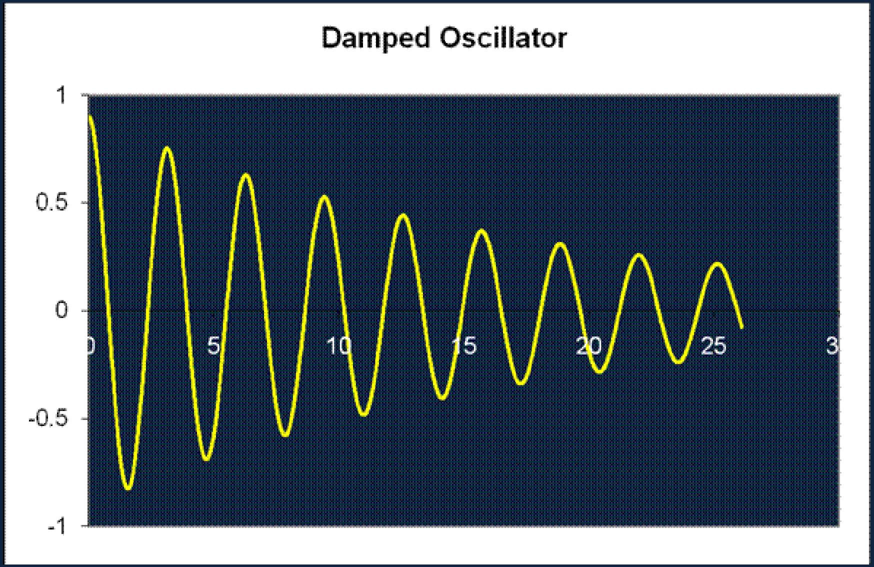
- The point to note here is that the damping can cause rapid decay of the oscillations without a perceptible change in the period (around 0.04% for $b = 0.11$, $k = 4$, $m = 1$).





Underdamped Motion

- Compare the curve with the equation: the successive position maxima follow an exponential curve $Ae^{-\gamma t}$, so any maximum reached is, say, 90% of the previous maximum.
- Remember the energy at maximum displacement is $\frac{1}{2}kx^2$.



$$x = Ae^{-\gamma t}\cos\omega' t$$

Clicker Question

- The amplitude in a damped oscillator reaches half its original value after four cycles. At which point does the oscillator have only half its original energy?

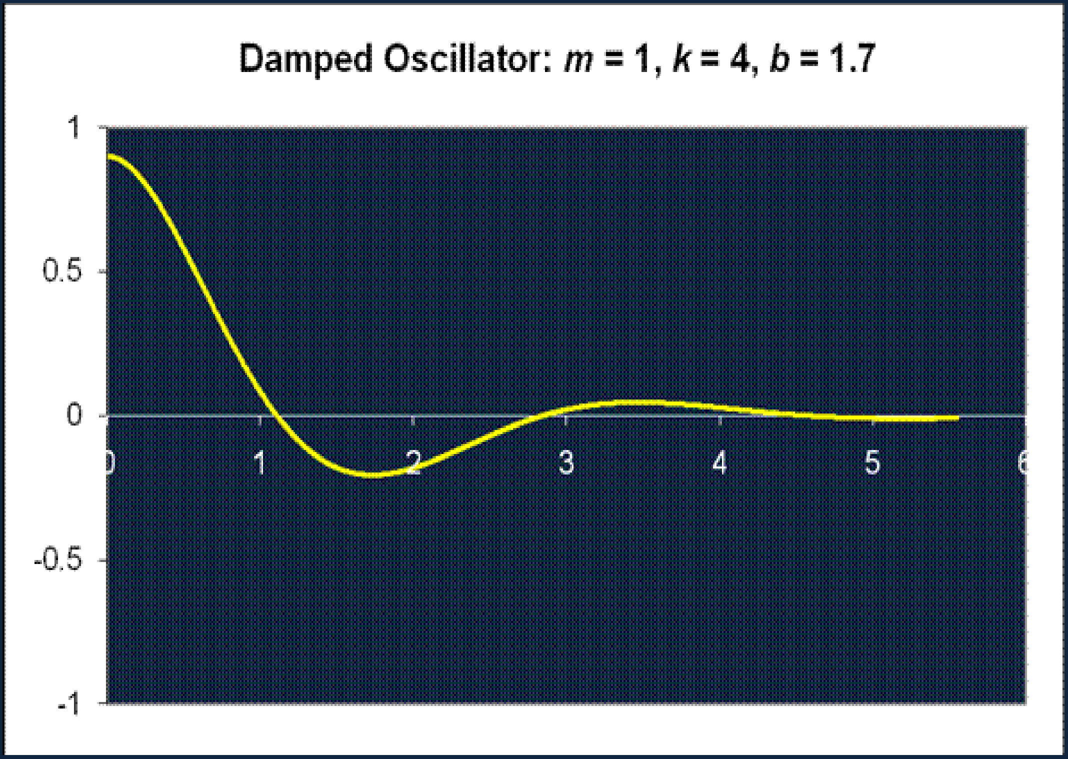
A. 2 cycles

B. 4 cycles

C. 8 cycles

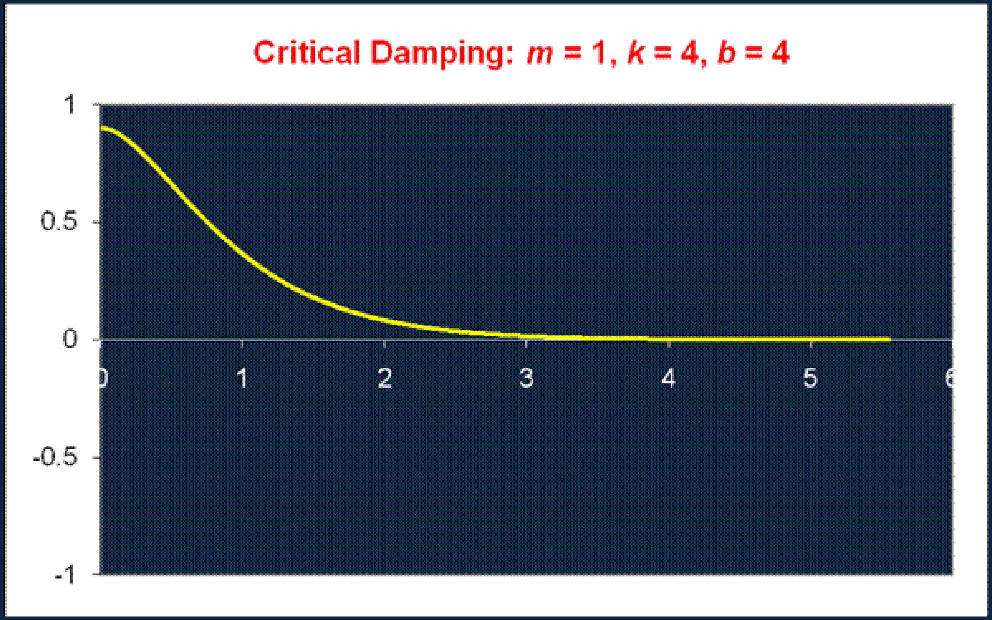
Not So Underdamped Motion

Even when the damping absorbs 98% of the energy in one period, the change in the length of the period is only around 10%!



Critical Damping

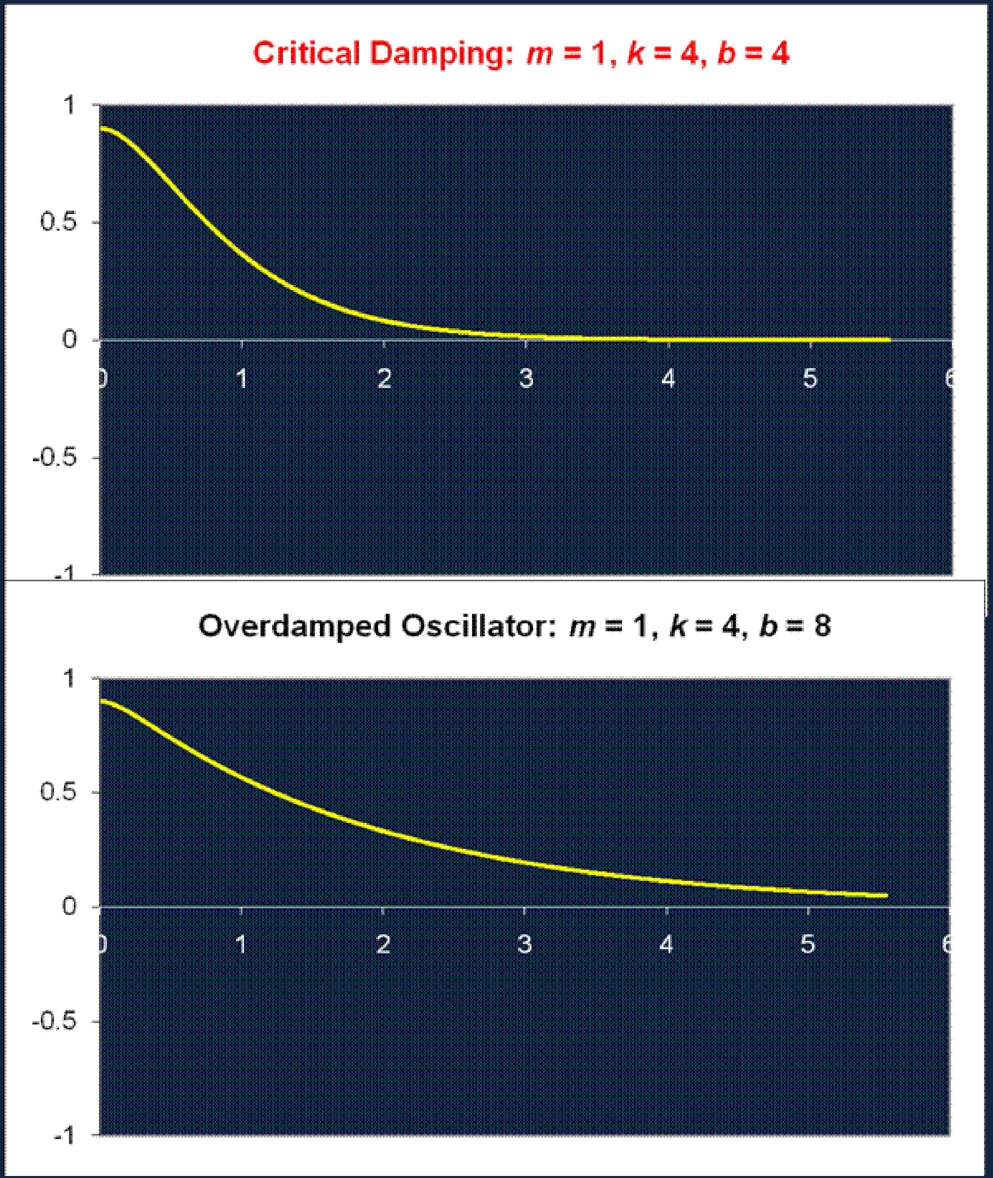
- As the damping is further increased, the period lengthens until at $b^2 = 4mk$ it becomes infinite, and the amplitude decays exponentially.



- (Actually, in this one case a prefactor $A + Bt$ is needed to match initial conditions—we'll ignore this minor refinement.)

Overdamping

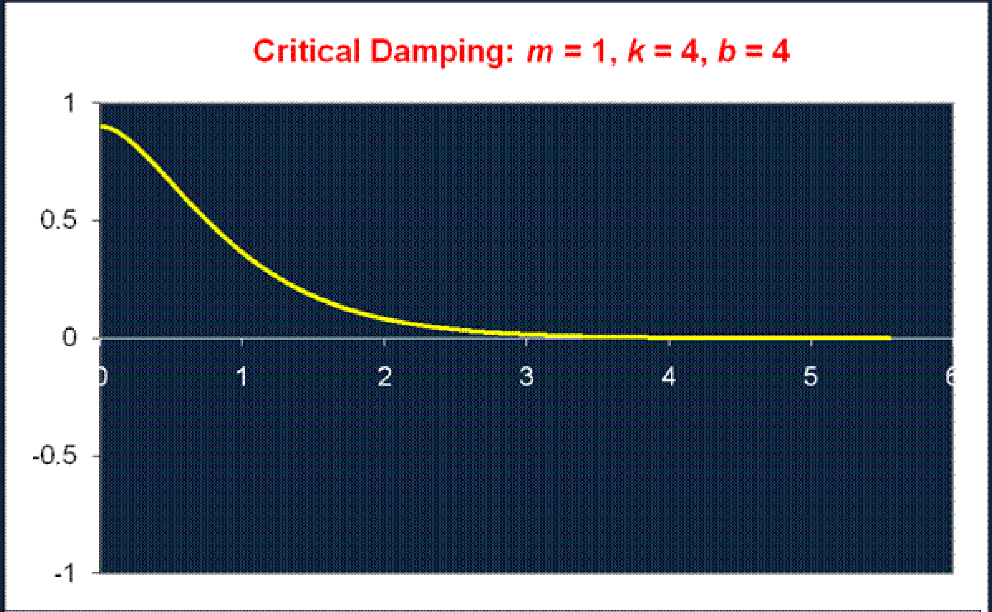
- Doubling the damping beyond critical damping just doubles the time for the amplitude to decay by a given amount.

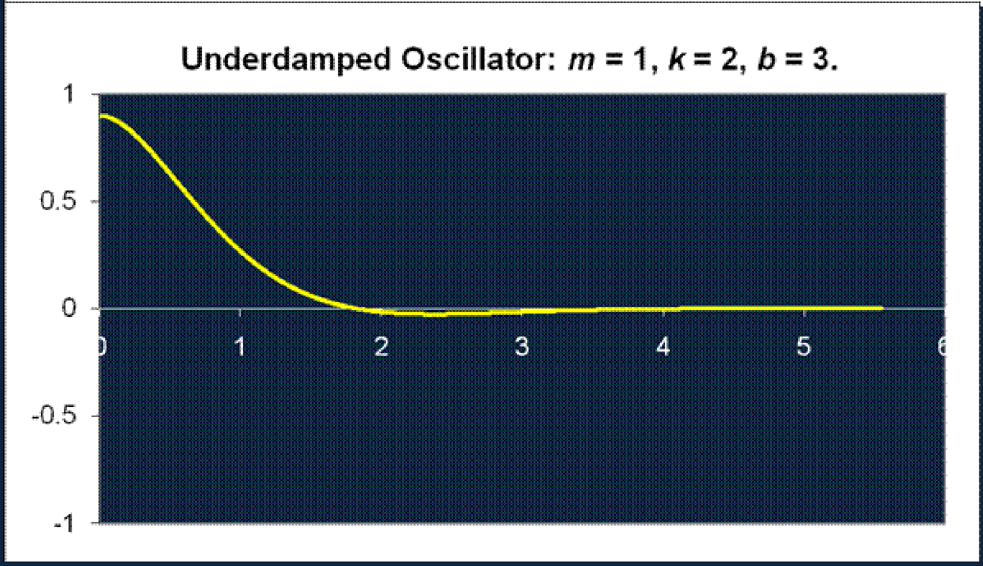


Ideal Damping for Shock Absorbers?

- Critical damping is not the best choice: underdamping gives a quicker response, and the overshoot can be very small.

- Explore this for yourself: download the spreadsheet!





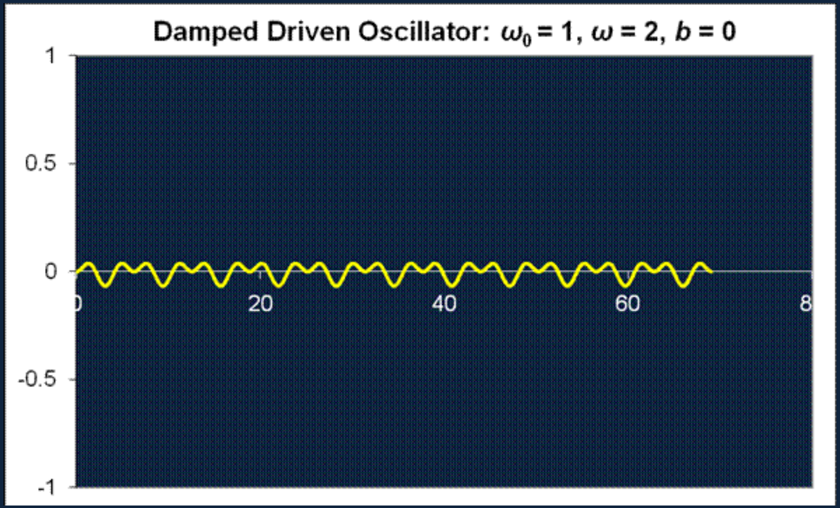
The Damped Driven Oscillator

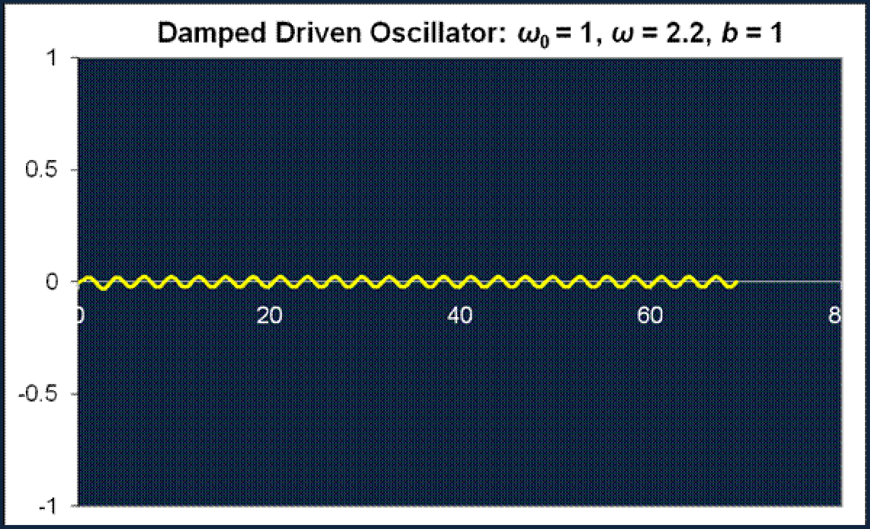
- We now consider a damped oscillator with an external harmonic driving force.
- We'll look at the case where the oscillator is well underdamped, and so will oscillate naturally at $\omega_0 = \sqrt{k / m}$.
- The external driving force is in general at a different frequency, the equation of motion is:

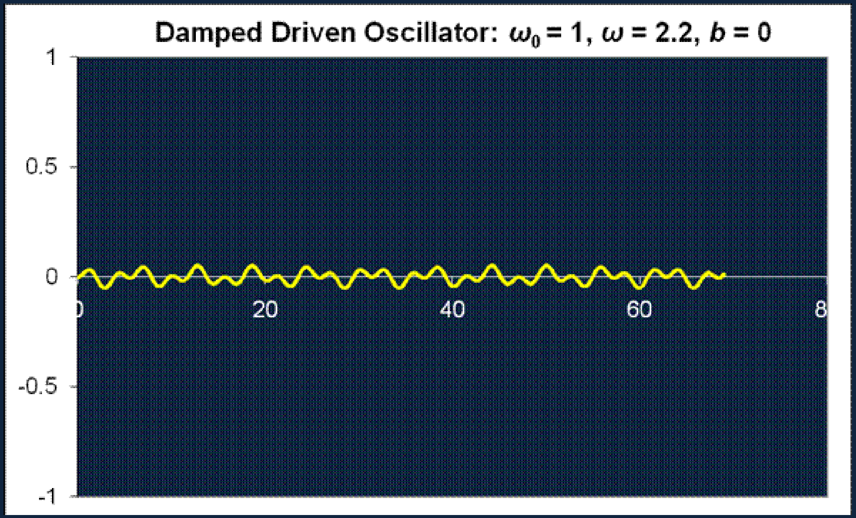
$$md^2x / dt^2 + bdx / dt + kx = F_0 \cos \omega t$$

The Damped Driven Oscillator

- If the driving frequency is far from the natural frequency, there is only a small response, even with no damping. Here the driving frequency is about twice the natural frequency.







The Damped Driven Oscillator

- This shows the oscillator with the same strength of external driving force, but at its natural frequency.
- The amplitude increases until damping energy losses equal external power input: this is **resonance**.
- Spreadsheet link!
- Tacoma Narrows Bridge.