Work and Energy

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Work: Lifting a Box

The word "work" as used in physics has a narrower meaning than it does in everyday life. First, it only refers to physical work, of course, and second, something has to be accomplished. Specifically, **a force does work on an object if the object moves—at least partially—in the direction the force is pushing.** Let's look at a couple of examples: If you lift up a box of books from the floor and put it on a shelf, you've done work, as defined in physics. But if the box proves too heavy, and you tug at it until you're worn out but it doesn't move, that doesn't count as work. Furthermore, carrying the box of books across the room from one shelf to another of equal height doesn't count either, because even though your arms had to exert a force upwards to keep the box from falling to the floor, you did not move the box in the direction of that force, that is, upwards. (This same horizontal displacement of the books from shelf to shelf could have been accomplished by the box sliding on a low friction surface at constant height, requiring essentially no effort.)

Think now of this vertical lifting of boxes as being done by a small machine, an electrically driven hoist, say, that doesn't wear out quickly as a human would. It's clear that to lift the box twice as high will take twice the work, and to raise a box twice as heavy will require twice the effort. In other words, the natural definition of work accomplished by a steady vertical force F raising a weight through a distance d is

$$\textbf{work} = \textbf{force} \times \textbf{displacement}$$

With this definition, the natural "unit of work" is the work done be a force of one newton pushing a distance of one meter. In other words (approximately) lifting a stick of butter three feet. *This unit of work is called one **joule***, in honor of an English brewer. (His important contribution—apart from good beer—was to establish that heat is a form of energy.)

$$\textbf{1 joule} = \textbf{1 newton} \times \textbf{1 meter}$$

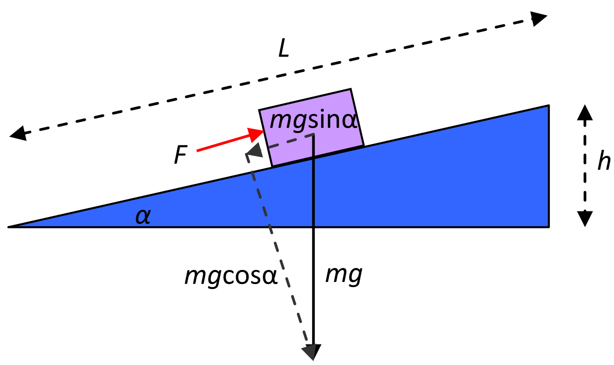
Back to the box of books: if it's lifted at a steady speed, the force F is just balancing off gravity, the weight mg of the box (otherwise the box would be accelerating: of course, initially a little more force would be needed to get it going, and then at the end a little less, as the box comes to rest at the height of the shelf.)

Bottom line: to raise a mass *m* vertically through a height *h* at a constant speed a force *mg* must be supplied, pointing upwards, and it will do work *mgh* in joules.

What about Sliding the Box up a Slope?

Suppose that instead of lifting the box directly through height *h* to get it on the shelf, we push it up a gentle slope, of length *L*, and let's make it smooth enough that we can ignore friction. Again, **we choose a force that just balances off gravity**, so the box progresses at a steady speed.

This force will be $F = mg\sin\alpha$, so the total work done $W = Lmg\sin\alpha = mgh$.



To push a box up a frictionless slope requires a force

$$F = mg\sin\alpha,$$

so the total work done

$$W = Lmg\sin\alpha = mgh.$$

Evidently the angle of the slope doesn't matter: as long as there's no friction, **the work done depends only on the height gained.** Obviously, the incline **doesn't even to be at constant slope,** because any curve can be approximated by a sequence of short stretches each at constant slope, and for each of these, the work done depends only on the height gained.

Of course, the other force on the box—gravity—must be doing work as well, the box is moving in the direction of a component of that force. But it's moving in the opposite direction to the way the force is pushing, so with our definition of work as force × displacement, **the work done by gravity is negative**! Gravity is actually absorbing work—storing it. If we let go of the box, as it slides back downhill this absorbed work is released: now gravity does the same amount of **positive** work on the box as we did in pushing it uphill.

Potential Energy

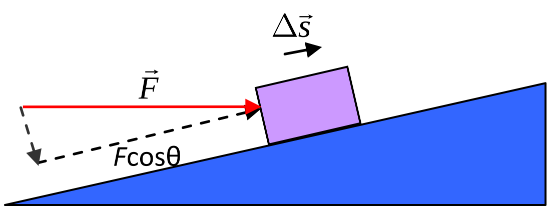
Basically, **the work we did in pushing the box uphill has been stored by gravity**, the amount is *mgh*, and this amount of work is available for future use: it is released by allowing gravity to act on the box as it comes back down. This "stored work" is called **potential energy**. It's equivalent to an amount of work, so naturally is measured in joules.

In this case, then

energy is the ability to do work

a formulation that unifies different kinds of energy, as we shall discover.

One more point: what if we push the box up the slope with a horizontal force, rather than a force parallel to the slope? It's clear from the definition of work that only the component of $\vec{F}$ in the direction of the movement does any work, as the box has no movement in the direction perpendicular to the surface. That is, $\Delta W = (F\cos\theta)\Delta s$.



The work done by a force $\vec{F}$ acting through a displacement $\Delta\vec{s}$ in a direction at an angle θ to $\vec{F}$,

$$\Delta W = F\Delta s\cos\theta$$

Expressing Work Done in Terms of the Scalar (or Dot) Product of Vectors

This combination of vector lengths and the cosine of the angle between them occurs so frequently in physics that it is convenient to have a special notation for it. It is called the **scalar product**, or **dot product**, of two vectors, and is written with a dot:

$$\vec{A}\cdot\vec{B} = AB\cos\theta$$

If we now consider an object tracing some path, with some variable force $\vec{F}$ acting on it throughout, as the body is displaced by $\Delta\vec{s}$, just as in the example above the force does work $\Delta W = \vec{F}\cdot\Delta\vec{s}$, so the total work done by the force going along the whole path is the sum of all these ΔW's,

$$W = \sum \vec{F}\cdot\Delta\vec{s} = \int \vec{F}\cdot\overrightarrow{ds}$$

writing it as an integral for the limit of taking the path increments small.

Storing Energy in a Spring

It's evident from the general expression for work done by a force along an arbitrary path that for gravity, a constant force downwards, W is simply proportional to the height difference between the ends of the path, as discussed above in the "box on a slope" section.

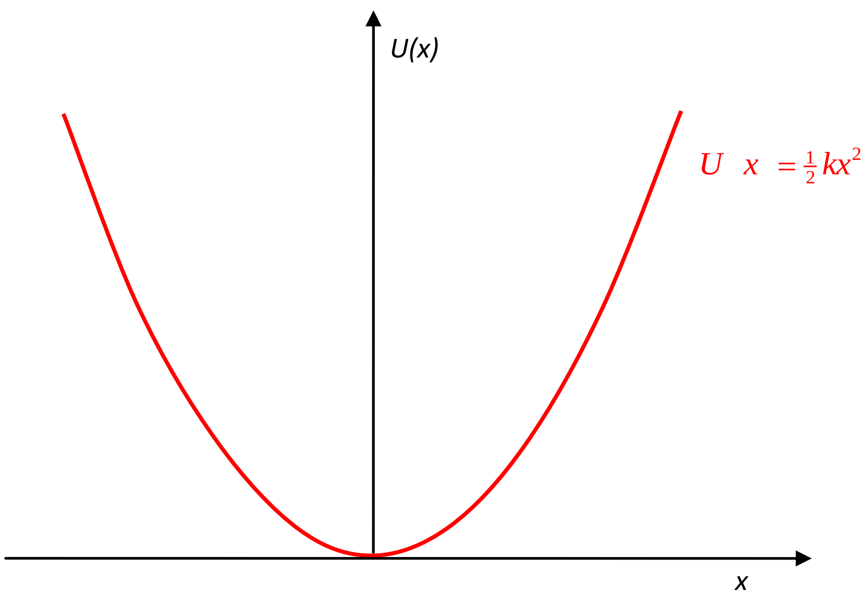
Consider now the work needed to stretch a spring. We'll restrict considerations of stretching and compressing to the "natural range" of the spring, meaning it goes back to its original length and shape when released. It is found experimentally that within this range springs obey Hooke's Law, the restoring force exerted by the spring is linearly proportional to the deviation x from the natural length, and of course in the opposite direction, back to the natural length:

$$F_{spring} = -kx$$

Therefore, to stretch a spring from its natural length L to $L + x_0$ takes work—we'll denote it by U, that is the standard notation for stored energy:

$$U(x_0) = \int \vec{F} \cdot \vec{ds} = \int_0^{x_0} kxdx = \tfrac{1}{2}kx_0^2.$$

This **potential energy** is available to do work as the spring goes back to its natural length. It's worth plotting potential energy as a function of extension x:



Potential Energy *U*(*x*) stored in a spring as a function of deviation from natural length.

Notice how the potential energy curve gets steeper and steeper as the extension increases: this is because the spring's restoring force increases with length, so each increment of distance takes more and more work.

The increase in potential energy ΔU on extending the spring from x to $x + \Delta x$ is:

$$\Delta U = F\Delta x = kx\Delta x$$

From which we see that the force the spring is exerting, $F = -kx$, is related to the potential energy by

$$F(x) = -dU(x)/dx.$$

Why the Minus Sign?

Getting the signs straight is a bit confusing at first: it must be clear whether *F* means the force applied to the spring from outside, or the force the spring itself is exerting on something else. The best way to handle the signs is to think through what's happening: if an outside force is doing work on the spring, the spring's potential energy is increasing, it's storing the work being done on it. If the spring is doing work on something else, the spring is depleting its store of potential energy.

This same equation connecting potential energy and force is clearly also true for gravitational potential energy, $U_{grav}(h) = mgh, \quad F_{grav} = -mg$, counting upwards as positive, and in fact is true in general: we shall soon be looking at the gravitational field well away from the Earth, and will generalize the expression for the gravitational potential energy to handle this more complicated situation, where $U = U(x, y, z)$ will vary in all three directions.

However, the variation of potential energy with position is still given by the work done on the system to move an incremental distance,

$$\Delta U(x, y, z) = \vec{F} \cdot \Delta\vec{s}$$

so the opposing force from the system itself is

$$\vec{F} = (F_x, F_y, F_z) = (-\partial U/\partial x, -\partial U/\partial y, -\partial U/\partial z).$$

These derivatives are *partial* derivatives: the component of $\vec{F}$ in the *x*-direction, F_x, only does work if there is a change in the *x*-coordinate, so how *U* changes in that direction relates to the force in that direction. The rule for partial differentiation is that if you're differentiating with respect to *x*, you treat *y* and *z* as if they're constants. Visually, that means you're finding the incremental change in potential energy on moving a small distance in the *x*-direction, so *y* and *z* don't change.

Energy: Kinetic and Potential

We know that resting a hammer head on a nail which is partly into a piece of hardwood accomplishes nothing. But if we drop the hammer on to the nail (and our aim is good!) it *will* do some work—the nail will be driven a little into the wood. We know from Newton's Laws that this happens because the moving hammer has momentum, and hitting the nail slows the hammer down rapidly—so the nail is exerting a large force on the hammer, and therefore the hammer exerts a large force on the nail, which drives it into the wood.

Let's analyze what's going on in the short interval of time between when the moving hammer first contacts the nail and the moment when the hammer comes to rest. We'll chose coordinates so the hammer is moving in the *x*-direction.

At some instant of time, suppose the hammer is exerting force F on the nail, so the nail is exerting $-F$ on the hammer. If during a tiny time interval, the nail moves distance dx, the work done by the hammer

$$dW = Fdx.$$

But at the same time, the nail's equal and opposite reaction force on the hammer is slowing it down,

$$-F = ma = m\frac{dv}{dt}$$

so

$$dW = -m\frac{dv}{dt}dx = -m\frac{dx}{dt}dv = -mvdv$$

Now integrate both sides over the period of the impact, to find the total work done on the nail by the hammer:

$$W = \int dW = -\int mvdv = -\left[\tfrac{1}{2}mv^2\right]_{v_o}^{0} = \tfrac{1}{2}mv_0^2$$

where v_0 is the speed of the hammer just before impact.

So the hammer is able to do this amount of work on the nail purely because the hammer is moving. Recall we defined energy as the ability to do work. We found a mass m raised to a height h had potential energy mgh, that's how much work it could deliver on going back to its original height. Now we've found that a mass m moving at speed v can deliver $\frac{1}{2}mv^2$ joules as it comes to rest. This is termed its kinetic energy—energy it has purely because it's moving.

Kinetic energy of mass m at speed $v = \frac{1}{2}mv^2$

And notice that our proof above doesn't depend on the deceleration being uniform—it probably isn't.

Conservation of Mechanical Energy

Consider now a block of mass m sliding down a frictionless slope of angle α.

The accelerating gravitational force down the plane is $F = mg\sin\alpha$, so when the block slides a small distance dx, gravity does work

$$dW = (mg\sin\alpha)dx = -mgdh.$$

This gravitational force is of course accelerating the block, $F = ma = mdv/dt$, so

$$dW = Fdx = m\frac{dv}{dt}dx = m\frac{dx}{dt}dv = mvdv = d\left(\tfrac{1}{2}mv^2\right).$$

We see that as the block moves through the incremental distance *dx*, its gravitational potential energy is depleted by an amount $dW = mgdh$, but at the same time its kinetic energy is increased by precisely the same amount.

That is to say, an amount of energy *dW* has been transferred from the block's potential energy to its kinetic energy.

We define the block's **total mechanical energy = potential energy + kinetic energy: this is constant** if there are **no frictional forces present**—just gravity and smooth surfaces.

Summarizing: $dW = -mgdh = d\left(\frac{1}{2}mv^2\right)$, so $d\left(\frac{1}{2}mv^2 + mgh\right) = 0$,

So if the block slides from h_0 to h_1,

$$\tfrac{1}{2}mv_0^2 + mgh_0 = \tfrac{1}{2}mv_1^2 + mgh_1.$$

This is the **conservation of total mechanical energy**. We've derived it for a block on a smooth slope, but our argument would be just as good for a frictionless roller coaster: we've established that for every little movement on a smooth slope energy is conserved, independent of the angle of slope or the direction, and a roller coaster can be represented as a large number of such slopes joined together.

Conservative and Non-Conservative Forces

The key to energy conservation with gravitational forces is that if work is done against gravity along some path, gravity stores that work as potential energy, it is not lost, and gravity will deliver it back to the object on a return path. The spring is the same way: work done to compress it is stored as potential energy, and releases as the spring expands. These are called conservative forces: they arise from a potential energy function, and as discussed previously the force is the negative of the differential of that function. This potential energy is then entered in the expression for the total energy of the system, along with kinetic energy, etc., and the total energy is conserved.

Well, actually, the total mechanical energy is not conserved if there is friction in the system. It takes work to drag a box horizontally across a rough floor, and friction doesn't store that work to help you drag the box back. Quite the contrary, it's just as hard to drag it the other way. So what happened to energy conservation? There is no doubt that total mechanical energy, as defined above, is not conserved. However, careful measurements reveal that heat is generated whenever there is friction. Heat is in fact a form of energy: it's kinetic and potential energy on a microscopic level, when something is hotter, the atoms are jiggling around faster. This can be dealt with quantitatively, as we'll see later in the course, and when it is we find that in fact total energy is conserved, but the total energy now includes the heat generated (not to mention the energy in the squeaky sounds emanating from the dragged box, etc.)

Work Done by a Conservative Force is Independent of Path Taken from A to B

This statement is obviously true of gravity: the work done by gravity is the change in potential energy along the path, and that's just $mg(h_B - h_A)$. But it must be true of any conservative force, for the following reason: suppose it's false, and the conservative force does more work on an object going from A to B via path P_1 that via path P_2. Then we can put a ball in a frictionless tube which goes from A to B along P_1, returns along P_2, and turns smoothly at both A and B to make a continuous tube. Starting from A, the force will speed up the ball as it goes to B, then lt will be slowed down on the P_2 return—but not so much! The slowing down will be just equal to the speeding up that would have occurred had the ball set out along P_2—and recall we said the force does more work, so delivers more energy, along P_1. This means that after one round trip, the ball is moving faster—and this will go on indefinitely, it will get faster and faster! This of course contradicts one of the basic postulates of physics and life, that you can't get something for nothing, there's no such thing as a perpetual motion machine. **So the work done by a conservative force is independent of path.**

Gravitational Potential Energy Far Above Earth

We've established that close to the Earth's surface, only vertical movement of an object counted as work done against gravity, because the incremental work is $\vec{F}\cdot\Delta\vec{s}$, and the force is vertically down, so along any path, the change in gravitational potential energy depends only on the net height difference.

When we move to heights comparable to the radius of the Earth, we can no longer take the gravitational force to be constant, it is of course

$$\vec{F} = -\frac{GM_E m}{r^2}\hat{\vec{r}}$$

where $\hat{\vec{r}}$ is a unit vector pointing radially outwards. Notice it's still vertically downwards, where now by vertically downwards we mean pointing to the center of the Earth. But this means that lifting something, even far above the Earth, only takes work for movement directly away from the center of the Earth—sideways motion doesn't count, just as at the Earth's surface. So the potential energy, the stored work, still depends only on height difference, but it's a bit more complicated because gravity gets weaker with height, so less effort is required as you get further away.

To find just how gravitational potential energy varies with height, all we need consider is a path going straight up. We have:

$$U(r_B) - U(r_A) = \int_{r_A}^{r_B} \frac{GM_E m}{r^2}dr = \left[-\frac{GM_E m}{r}\right]_{r_A}^{r_B} = \frac{GM_E m}{r_A} - \frac{GM_E m}{r_B}.$$

It follows that

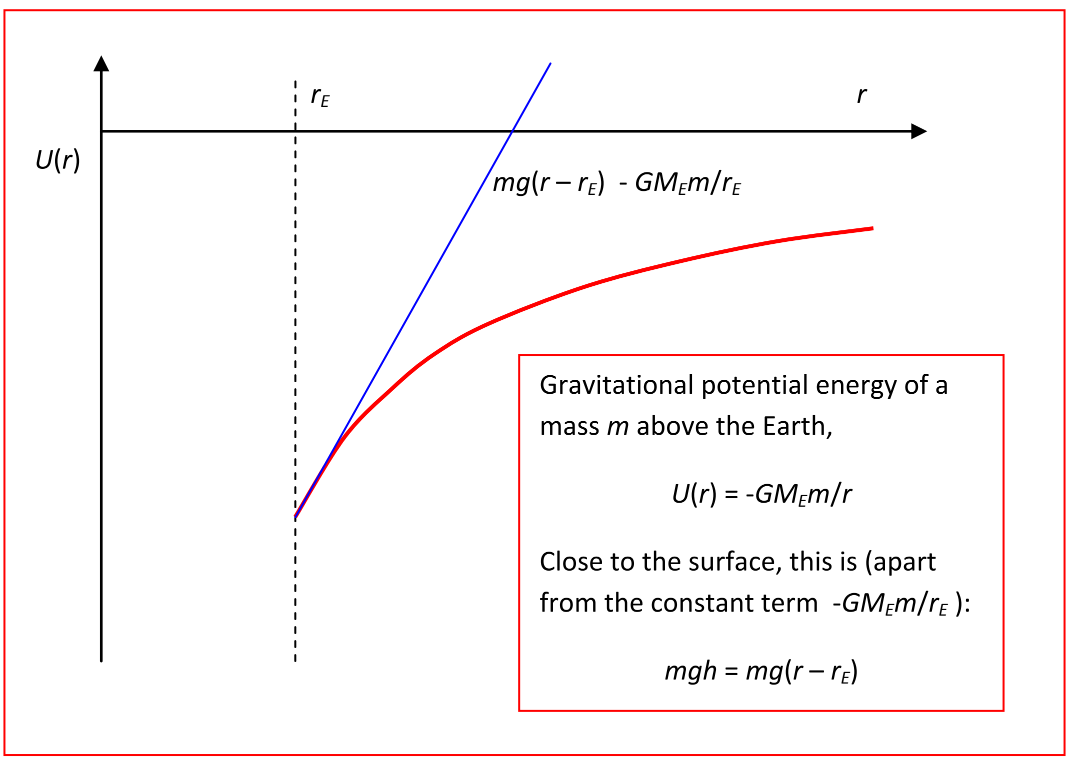
$$U(r) = -\frac{GM_E m}{r} + C,$$

where C is a constant of integration, which has to be determined some other way. In a sense, this constant really doesn't matter, because we only ever work with *changes* in potential energy. In considering gravitational potential energy near the Earth's surface, we wrote mgh where h was measured from the floor, say. But that is an arbitrary choice, and in going to the basement the potential energy will of course be negative. To avoid messing with signs, it's usually convenient to take the zero below any level reached in the situation under discussion. To repeat: only changes in potential energy have any significance, so it really doesn't matter where you choose zero to be.

Things look a little different, though, on an astronomical scale. This potential energy is what an object has by virtue of its interaction with the Earth's gravitational field. But what about a star in a distant galaxy? Obviously, the Earth's gravitational field has zero impact on its existence. To take account of this in our definition, we need the potential energy in the Earth's field to go to zero at large distances—meaning we should drop the C, and just write

$$U(r) = -\frac{GM_E m}{r}.$$

Here's a picture:



To get used to the idea of negative potential energy, imagine an object far from Earth (say a million miles) moving very slowly towards it. The object's total energy—kinetic plus potential—will be close to zero. As it falls to Earth, its total energy doesn't change: it picks up speed, gaining kinetic energy, but its potential energy is increasingly negative—in the picture above, it's sliding down the hill.

Escape Velocity and Orbital Velocity

For a spacecraft starting from the Earth to escape entirely, it must have enough kinetic energy to climb all the way up the hill in the picture above. In other words, it must have total energy zero! That is to say, we need

$$\tfrac{1}{2}mv_{\text{esc}}^2 = \frac{GM_E m}{r_E}$$

with r_E the Earth's radius. It's straightforward to check that v = 11 km/sec, approximately.

It's interesting to compare this equation with the equation of motion for a circular orbit at the same r. (Real orbiting satellites of course have r greater than r_E, but for the low ones it's a small correction.)

$F = ma$ for a circular orbit at r_E is

$$\frac{mv_{\text{orbit}}^2}{r_E} = \frac{GM_E m}{r_E^2}$$

This is almost the same equation! It's easy to check by comparing the two that there is a simple relationship between the circular orbital speed and the escape velocity:

$$v_{\text{esc}} = \sqrt{2}v_{\text{orbit}}.$$

Power

From a practical engineering point of view, it's obviously essential to know how many boxes an hour a hoist can raise, or how soon a given pump can get all the water out of the basement. We need a unit for rate of working, also known as power.

The SI unit of power is one joule per second, and this is called one *watt*.

The name commemorates James Watt, the engineer who constructed the first really useful steam engine, in the 1760's. He invented his own unit of power: the horsepower, to rate his engines. The horsepower is 746 watts. A good horse can work through the day at about 0.7 horsepower, say 500 watts (or half a kilowatt).

What's your personal wattage? Consider walking (or running) upstairs. A typical step is eight inches, or one-fifth of a meter, so walking upstairs at a reasonable pace you will gain altitude at, say, three-fifths of a meter per second. Your weight is (put in your own weight here!) 70 kg. (for me) multiplied by 10 to get it in newtons, 700 newtons. The rate of working then is 700 x 3/5, or 420 watts. You could probably double your power by running up, but not for long! To get an idea of possible human power, world champion cyclist Lance Armstrong put out 400 watts for several hours (few people can!). Ordinary recreational cycling is in the 100 - 150 watt range.