Newton's Law of Gravity and Kepler's Laws

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These notes are partly adapted from my Physics 152 lectures, where more mathematical details can be found.

The Universal Law of Gravitation

Newton boldly extrapolated from the earth, the apple and the moon to everything, asserting that every body in the universe attracted every other body with a gravitational force that decreased with distance as $1/r^2$: a Universal Law of Gravitation.

But actually he knew more about the gravitational force: from the fact that bodies of different masses near the earth's surface accelerate downwards at the same rate, using $F = ma$ (his Second Law) if two bodies of different masses have the same acceleration they must be feeling forces in the same ratio as their masses (so a body twice as massive feels twice the gravitational force), that is, *the gravitational force of attraction a body feels must be proportional to its mass*. Now suppose we are considering the gravitational attraction between two bodies (as we always are), one of mass m_1, one of mass m_2. By Newton's Third Law, the force body 1 feels from 2 is equal in magnitude (but of course opposite in direction) to that 2 feels from 1. If we think of m_1 as the earth, the force m_2 feels is proportional to m_2, as argued above—so this must be true whatever m_1 is. And, since the situation is perfectly symmetrical, *the force must also be proportional to* m_1.

Putting all this together, the magnitude of the gravitational force between two bodies of masses m_1 and m_2 a distance r apart

$$F = Gm_1m_2/r^2.$$

The constant $G = 6.67\times10^{-11}\ N.m^2/kg^2$—we'll discuss how G is measured shortly.

You might be wondering: what, exactly, is this distance r? It seems pretty clear for the Earth and the Moon, but what about the Earth and a person standing on its surface? Is r zero? What does that mean?

The answer is that ***r* is measured from the center of the Earth.**

Newton proved that too: he interpreted his Law of Gravitation as meaning there is an inverse square attraction between the person and every separate bit of the Earth. He then envisioned the Earth as made up of spherical shells, one inside the next, like an onion, and proved by doing an integral that **the gravitational attraction from a uniform hollow spherical shell of mass *m* is the same as if all the mass were concentrated at the center.** (*Note*: this is the gravitational attraction *outside* the shell, we'll discuss what happens inside the shell in the next section.)

It then follows that the same is true of the Earth, a collection of spherical shells all having the same center.

This means g at the surface of the Earth is given by

$$g = Gm_E/r_E^2$$

where m_E is the mass and r_E the radius of the Earth.

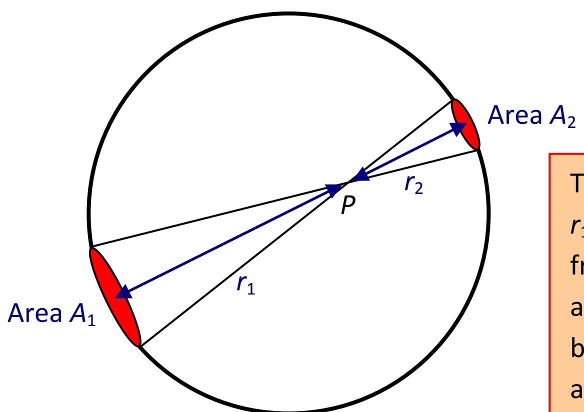
Notice that by measuring g, and knowing r_E, we *can* find Gm_E. But this does *not* tell us what G is, since we don't know m_E! It turns out that this same problem arises with every astronomical observation. Timing the planets around the Sun will give us Gm_{Sun}. So we *can* figure out the ratio of the Sun's mass to the Earth's, but we can't find an absolute value for either one.

The first measurement of G was made in 1798 by Cavendish, a century after Newton's work. Cavendish measured the tiny attractive force between lead spheres of known mass. For details on how an experiment at the University of Virginia in 1969 improved on Cavendish's work, click on the UVa Physics site here.

Cavendish said he was "*weighing the Earth*" because once G was measured, he could immediately find the mass of the Earth m_E from $g = Gm_E/r_E^2$, and then go on the find the mass of the Sun, etc.

Field *Inside* a Spherical Shell

What about the gravitational attraction felt *inside* a uniform spherical shell of mass? This is important to know, because if you visualize the Earth as made up of spherical shells, when you go down a mine you're inside some of the shells.



The ratio of the two areas $A_1/A_2 = r_1^2/r_2^2$, so the gravitational forces from these two areas *cancel* at P—and the whole spherical shell can be divided into opposite cancelling areas, meaning **the total gravitational force at *P* from the whole shell is zero.**

Finding the force inside turns out to be surprisingly simple! Imagine the shell to be very thin, with a mass density ρ kg per square meter of surface. Begin by drawing a two-way cone radiating out from the point P, so that it includes two small areas of the shell on opposite sides: these two areas will exert gravitational attraction on a mass at P in opposite directions. It turns out that they exactly *cancel*.

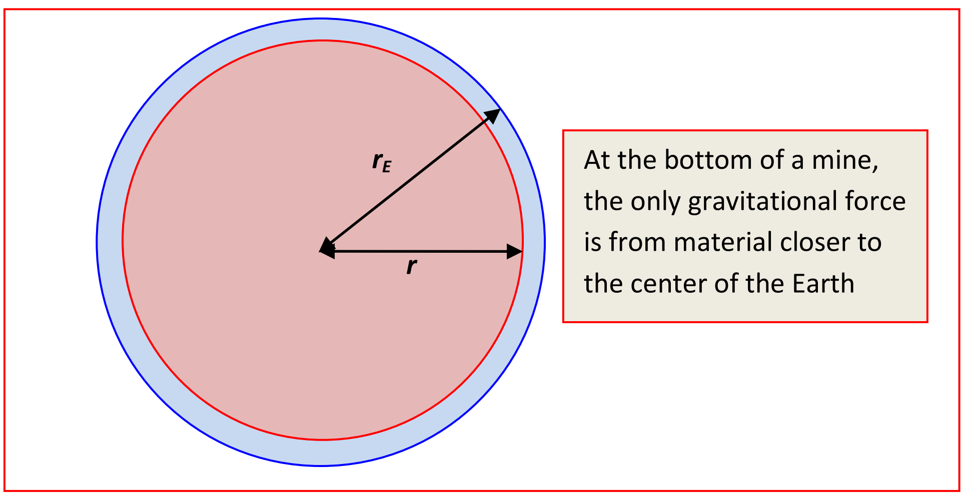
This is because the ratio of the areas A_1 and A_2 at distances r_1 and r_2 are given by $A_1 / A_2 = r_1^2 / r_2^2$: since the cones have the same angle, if one cone has twice the height of the other, its base will have twice the diameter, and therefore *four times the area*. Since the masses of the bits of the shell are proportional to the areas, the ratio of the masses of the cone bases is also r_1^2 / r_2^2. But the gravitational attraction at *P* from these masses goes as Gm / r^2, and that r^2 term *cancels* the one in the areas, so the two opposite areas have equal and opposite gravitational forces at *P*.

In fact, the gravitational pull from every small part of the shell is balanced by a part on the opposite side—you just have to construct a lot of cones going through *P* to see this. (There is one slightly tricky point—the line from *P* to the sphere's surface will in general cut the surface at an angle. However, it will cut the opposite bit of sphere *at the same angle*, because any line passing through a sphere hits the two surfaces at the same angle, so the effects balance, and the base areas of the two opposite small cones are *still* in the ratio of the squares of the distances r_1, r_2.)

Field Inside a Solid Sphere: How Does *g* Vary on Going Down a Mine?

This is a practical application of the results for shells. On going down a mine, if we imagine the Earth to be made up of shells, we will be inside a shell of thickness equal to the depth of the mine, so will feel *no* net gravity from that part of the Earth. However, we will be *closer* to the remaining shells, so the force from them will be intensified.

Suppose we descend from the Earth's radius r_E to a point distance r from the center of the Earth. What fraction of the Earth's mass is still attracting us towards the center? Let's make life simple for now and assume the Earth's density is *uniform*, call it ρ kg per cubic meter.



Then the fraction of the Earth's mass that is still attracting us (because it's closer to the center than we are—inside the red sphere in the diagram) is $V_{red} / V_{blue} = \frac{4}{3}\pi r^3 / \frac{4}{3}\pi r_E{}^3 = r^3 / r_E^3$.

The gravitational attraction from this mass at the bottom of the mine, distance r from the center of the Earth, is proportional to mass/r^2. We have just seen that the mass is itself proportional to r^3, so the actual gravitational force felt must be proportional to $r^3/r^2 = r$.

That is to say, the gravitational force on going down inside the Earth is *linearly proportional to distance from the center*. Since we already know that the gravitational force on a mass m at the Earth's surface $r = r_E$ is mg, it follows immediately that in the mine the gravitational force must be

$$F = mgr/r_E .$$

So there's no force at all at the center of the Earth—as we would expect, the masses are attracting equally in all directions.

Describing the Solar System: Kepler's Laws

Newton's first clue that gravitation between bodies fell as the inverse-square of the distance may have come from comparing a falling apple to the falling Moon, but important support for his idea was provided by a detailed description of planetary orbits constructed half a century earlier by Johannes Kepler.

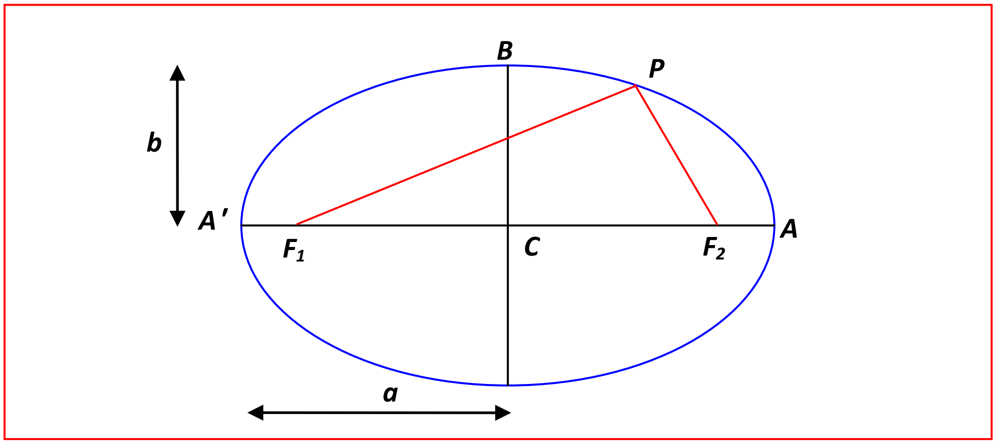
Kepler had inherited from Tycho Brahe a huge set of precise observations of planetary motions across the sky, spanning decades. Kepler himself spent eight years mathematically analyzing the observations of the motion of Mars, before realizing that Mars was moving in an *elliptical* path.

To appreciate fully how Kepler's discovery confirmed Newton's theory, it is worthwhile to review some basic properties of ellipses.

Ellipses

A *circle* can be defined as the set of all points which are the same distance R from a given point, so a circle of radius 1 centered at the origin O is the set of all points distance 1 from O.

An *ellipse* can be defined as the set of all points such that the *sum* of the distances from two fixed points is a constant length (which must obviously be greater than the distance between the two points!). This is sometimes called the gardener's definition: to set the outline of an elliptic flower bed in a lawn, a gardener would drive in two stakes, tie a loose rope between them, then pull the rope tight in all different directions to form the outline.

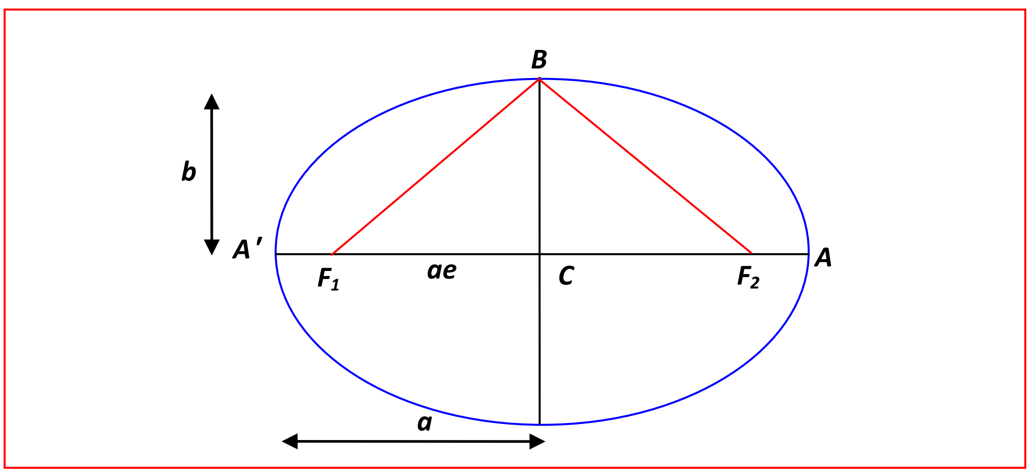


In the diagram, the stakes are at F_1, F_2, the red lines are the rope, P is on the ellipse.

CA is called the semimajor axis length a, *CB* the semiminor axis, length b. F_1, F_2 are called the foci (plural of focus).

Notice first that **the string has to be of length 2*a***, because it must stretch along the major axis from F_1 to A then back to F_2, so there's a double length of string along F_1A and a single length from F_1 to F_2. But the length $A'F_1$ is the same as F_2A, so the total length of string is the same as the total length $A'A = 2a$.

Suppose now we put P at B. Since $F_1B = BF_2$, and the string has length $2a$, **the length $F_1B = a$**.



Applying Pythagoras' theorem to the triangle F_1BC,

$$F_1C^{\ 2} = a^2 - b^2.$$

We shall use this result shortly.

Evidently, for a circle, $F_1C = 0$. The *ellipticity* of the ellipse is defined as the ratio of F_1C to a,

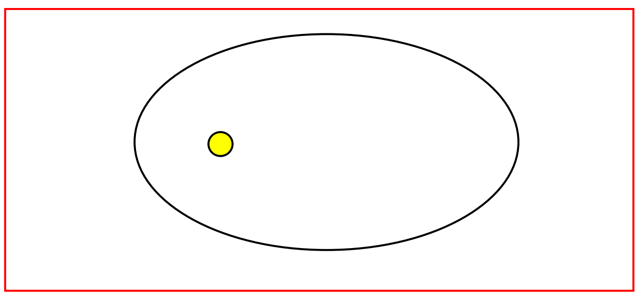
so

$$\text{ellipticity } e = F_1C / a = \sqrt{1-(b/a)^2}$$

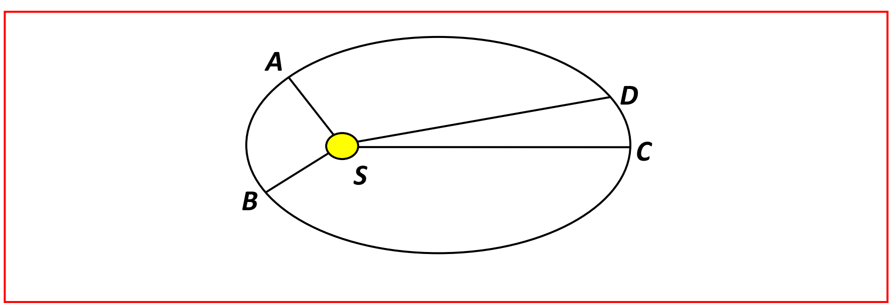
F_1 and F_2 on the diagram are called the *foci* of the ellipse (plural of *focus*) because if a point source of light is placed at F_1, and the ellipse is a mirror, it will reflect—and therefore *focus*—all the light to F_2.

Kepler summarized his findings about the solar system in three laws:

1. The planets all move in elliptical orbits with the Sun at one focus.



2. As a planet moves in its orbit, the line from the center of the Sun to the center of the planet sweeps out equal areas in equal times, so if the area *SAB* (with curved side *AB*) equals the area *SCD*, the planet takes the same time to move from *A* to *B* as it does from *C* to *D*.



For my Flashlet illustrating this law, click here.

3. The time it takes a planet to make one complete orbit around the sun *T* (one planet year) is related to its average distance from the sun *R*:

$$T^2/R^3 = \text{constant,}$$

the same constant for all planets orbiting the Sun.

In other words, if a table is made of the length of year *T* for each planet in the Solar System, and its average distance from the Sun *R*, and T^2/R^3 is computed for each planet, the numbers are all the same.

These laws of Kepler's are precise, but they are only *descriptive*—Kepler did not understand why the planets should behave in this way. Newton's great achievement was to prove that all this complicated behavior was the consequence of his one simple law of attraction.

How Newton Used his Law of Universal Gravitation to Explain Kepler's Laws

Kepler's First Law: the Planets Move in Elliptical Paths

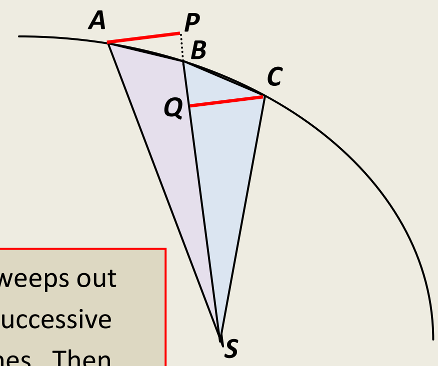
Surprisingly, the first of Kepler's laws is the toughest to prove beginning with Newton's assumption of inverse-square gravitation. Newton himself did it with an ingenious geometrical argument, famously difficult to follow. It can be more easily proved using calculus—if you want to see the proof, it's here.

Kepler's Second Law: Equal Areas in Equal Times

Newton proved from Kepler's Second Law that the orbiting planet felt a gravitational force directed towards the Sun.

This now seems obvious, but Kepler himself thought the planets were pushed around their orbits by invisible spokes radiating out from the Sun. (He thought a push was needed to keep something moving.)

According to Kepler's Second Law, if the planet in the diagram below goes from *A* to *B* in a month, then from *B* to *C* in the next month, the areas of the triangles *ABS* and *BCS* are equal. (Actually, it's the area inside the orbit, which is very slightly different, but taking shorter time intervals these areas become the same, as is clear in the figure.)



Kepler's Second Law: a planet in orbit sweeps out equal areas in equal times. *A*, *B*, *C* are successive points in the orbit, *AB*, *BC* take equal times. Then area *SAB* = area *SBC* (*S* = Sun)

Now the area of a triangle is ½ base × height. The two triangles *ABS*, *BCS* have a common base *BS*, so since they're observed to have equal areas, they must also have the same height as measured from that base. In the figure, this means the two lines *QC* , *AP* have equal length,

$$QC = AP.$$

In the first month, the planet goes from *A* to *B*, so the line *AB* is a vector representing its average velocity in km per month during that month, and similarly the line *BC* represents its velocity during the second month.

These two velocity vectors can be resolved into components parallel to the direction from *B* to the Sun, that is along *BS*, and perpendicular to it. This resolution into components is represented in the diagram by the triangles *ABP*, *BCQ*. Now since *AP* = *QC*, the velocity component in the direction *BS* doesn't change: but in the direction along *BS*, the velocity component increases from *PB* to *BQ*.

The bottom line is that Kepler's Second Law proves the acceleration of the planet in orbit is *exactly* towards the Sun: so the force on the planet is in that direction.

(Note for experts: it might appear at first glance that the constancy to leading order of the velocity component perpendicular to BS is not consistent with angular momentum conservation, since the distance to the Sun is changing. However, AP and QC are not *perpendicular to the radial vectors at the midpoints of the intervals AB, BC.)*

Kepler's Third Law: T^2/R^3 = constant

Newton proved from this Law that the attraction towards the Sun was *inverse square*.

We shall confine ourselves to the case circular orbits, because the planetary orbits are in fact very close to circles, and this makes the math a lot easier! (Newton of course did it for ellipses.)

The acceleration of a planet moving at speed v in a circular orbit of radius R is v^2/R towards the center, so $\vec{F} = m\vec{a}$ is:

$$\frac{mv^2}{R} = G\frac{Mm}{R^2}.$$

The time for one orbit (the year) is $T = 2\pi R/v$, so dividing both sides of the equation above by R, we find:

$$\left(\frac{T}{2\pi}\right)^2 = \frac{R^3}{GM}, \quad \text{so } \frac{T^2}{R^3} = \frac{4\pi^2}{GM}.$$

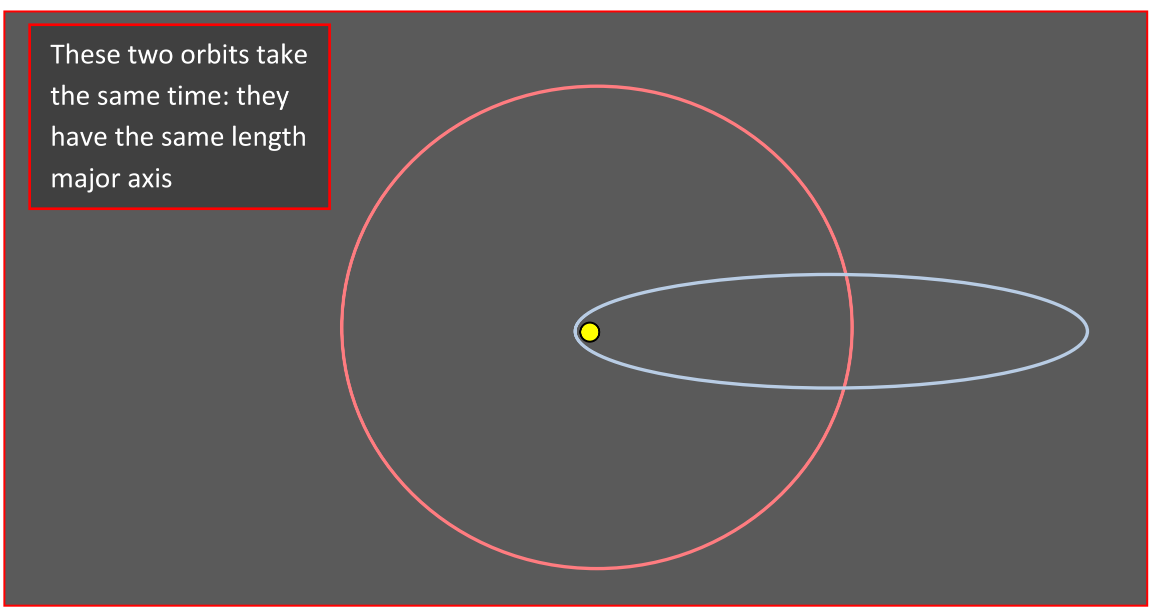
This is Kepler's Third Law: T^2 / R^3 is the same number for all the planets: it depends only on G and the mass of the Sun M.

Exercise: how are R, T related if the gravitational force is proportional to $1/R$? to $1/R^3$? To R?

The point of the exercise is that Kepler's Third Law, based on observation, forces us to the conclusion that the Law of Gravity is indeed inverse square.

Newton established that for the correct (elliptical) orbits, R should be be replaced by a, the semimajor axis of the ellipse, that is to say T^2 / a^3 is the same for all planets. This is in fact precisely what Kepler had discovered to be true.

It follows immediately that all elliptic orbits with the same major axis length, whatever their ellipticity, have the same orbital time.



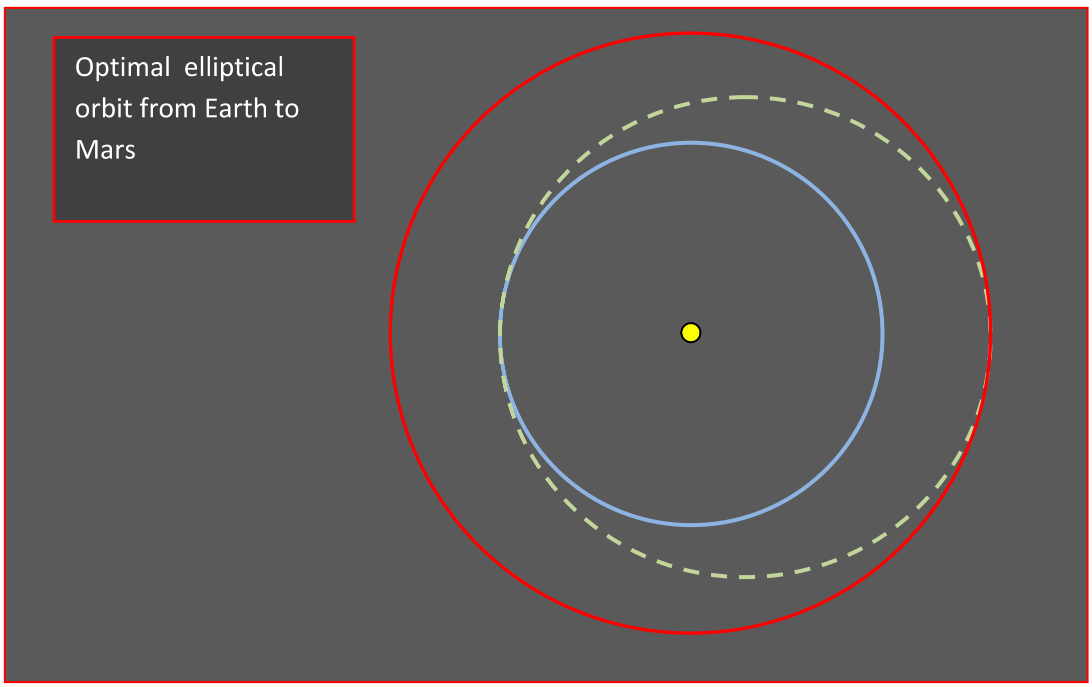
Note: a planet sweeps out area at a constant rate, but that rate is *not* the same for different planets!

Visiting the Planets: a Trip to Mars

To get to Mars with the least expenditure, we need to figure out how to blast off from the Earth in an orbit that just makes it out as far as Mars. We don't want to be firing the rocket engines during the trip, except very briefly to make small orbital corrections. This means that for almost the whole trip our ship will be following an elliptical orbit, just like a planet or asteroid. To get as much initial speed as possible for a given amount of fuel, we need to maximize the benefit of the Earth's own orbital motion (about 30 km/sec), so we'll blast off in the same direction. Our orbit will then edge outwards from the Earth's (approximately circular) orbit, since we're moving faster, just like Newton's cannon if fired at greater than the speed required for a circular orbit.

Since we set off with a velocity parallel to the Earth's, and move outwards, our elliptical orbit must touch the Earth's circular orbit at the beginning. We just need the ellipse to go outwards as far as the orbit of Mars—anything further is a waste of fuel.

Therefore, the orbit is an ellipse touching the Earth's circular orbit at its near point to the Sun, and Mars's orbit (which we also take circular) at the furthest point from the Sun:



Try your skill at getting to Mars here.

You'll find the appropriate launch speed is a little over 3 km/sec relative to Earth—once you're far enough away that Earth's gravitational field can be neglected, of course. (It isn't factored in here.)

Really Getting Out There: the Jupiter Slingshot

It is necessary to use the gravitational pull of other planets to get to the outer Solar System. It is at first sight surprising that this is possible, certainly the planet speeds up the spaceship as it approaches, but doesn't it equally slow it down as it moves on? The answer is yes, in the planet's own frame of reference, but the planet is moving. A ball hitting a tennis racquet during a serve leaves the racquet at the same speed relative to the racquet with which it hit it (approximately), and the same principle applies here. Check out this flashlet on the slingshot: the insert shows the expected trajectory relative to Jupiter, which you can see is symmetrical, meaning that it goes outward relative to Jupiter with the same speed it came in, for the same distance away. However, Jupiter's own speed, about 13 km/sec, must be added in to see what happens in our frame of reference.

We should mention that the trajectory in Jupiter's frame is a hyperbola rather than an ellipse, since it has sufficient speed to escape Jupiter's gravity entirely. We shall return to this case when discussing energy.

Many more mathematical details of the different orbits, and related topics not covered here such as the tides and general relativity, can be found in my Physics 152 Notes.