Electric Potential II

Physics 2415 Lecture 7

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Today's Topics

- Field lines and equipotentials
- Partial derivatives
- Potential along a line from two charges
- Electric breakdown of air

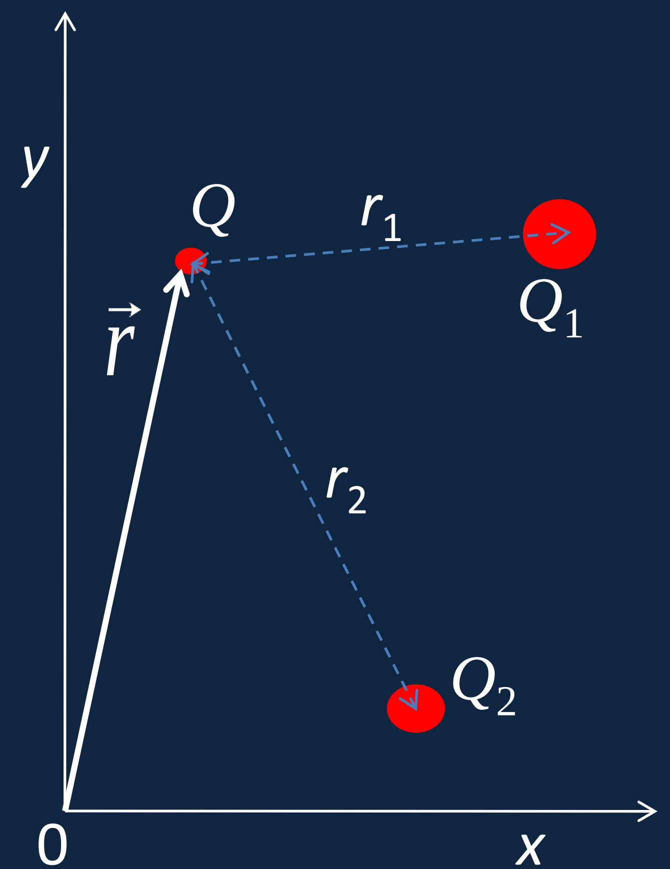
Potential Energies Just Add

- Suppose you want to bring one charge Q close to two other fixed charges: Q_1 and Q_2.
- The electric field Q feels is the sum of the two fields from Q_1, Q_2, the work done in moving $\overrightarrow{d\ell}$ is

$$\vec{E}\cdot\overrightarrow{d\ell}=\vec{E}_1\cdot\overrightarrow{d\ell}+\vec{E}_2\cdot\overrightarrow{d\ell}$$

so since the potential energy change along a path is work done,

$$V(\vec{r})=V_1(\vec{r})+V_2(\vec{r})$$



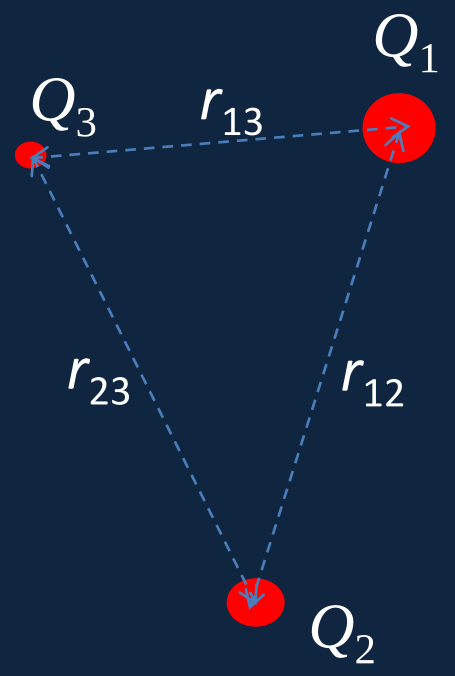
$$V(\vec{r})=\frac{1}{4\pi\varepsilon_0}\left(\frac{Q_1}{r_1}+\frac{Q_2}{r_2}\right)$$

Total Potential Energy: Just Add Pairs

- If we begin with three charges Q_1, Q_2 and Q_3 initially far apart from each other, and bring them closer together, the work done—the potential energy stored—is

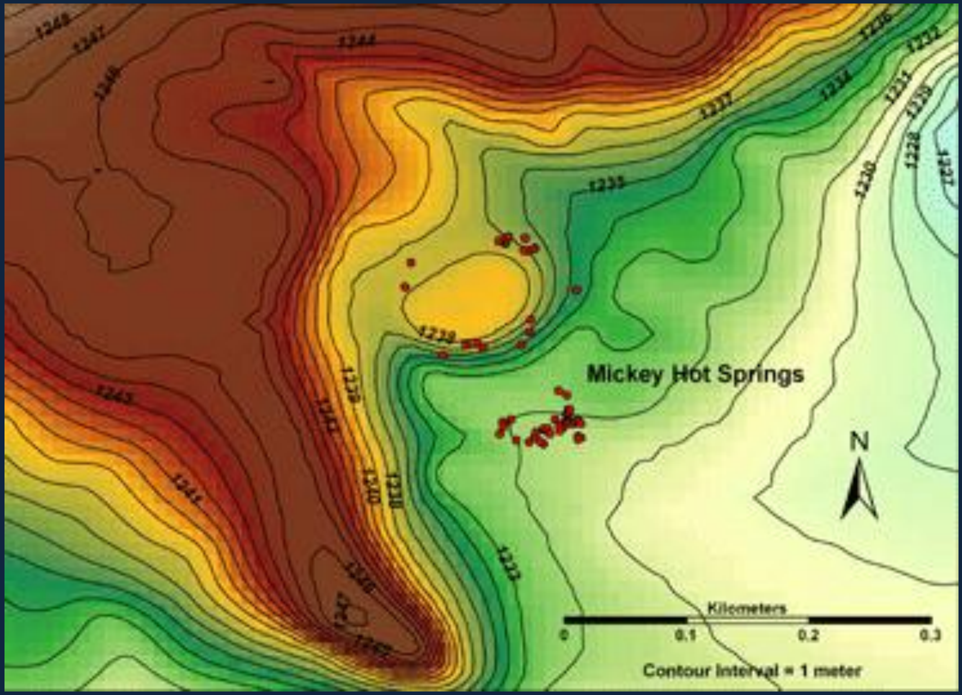
$$U = \frac{1}{4\pi\varepsilon_0}\left(\frac{Q_1Q_2}{r_{12}} + \frac{Q_2Q_3}{r_{23}} + \frac{Q_3Q_1}{r_{31}}\right)$$

and the same formula works for assembling any number of charges, just add the PE's from all pairs—avoiding double counting!



Equipotentials

- Gravitational equipotentials are just contour lines: lines connecting points (x,y) at the same height. (Remember PE = mgh.)
- It takes no work against gravity to move along a contour line.
- *Question*: What is the significance of contour lines crowding together?



Electric Equipotentials: Point Charge

- The potential from a point charge Q is

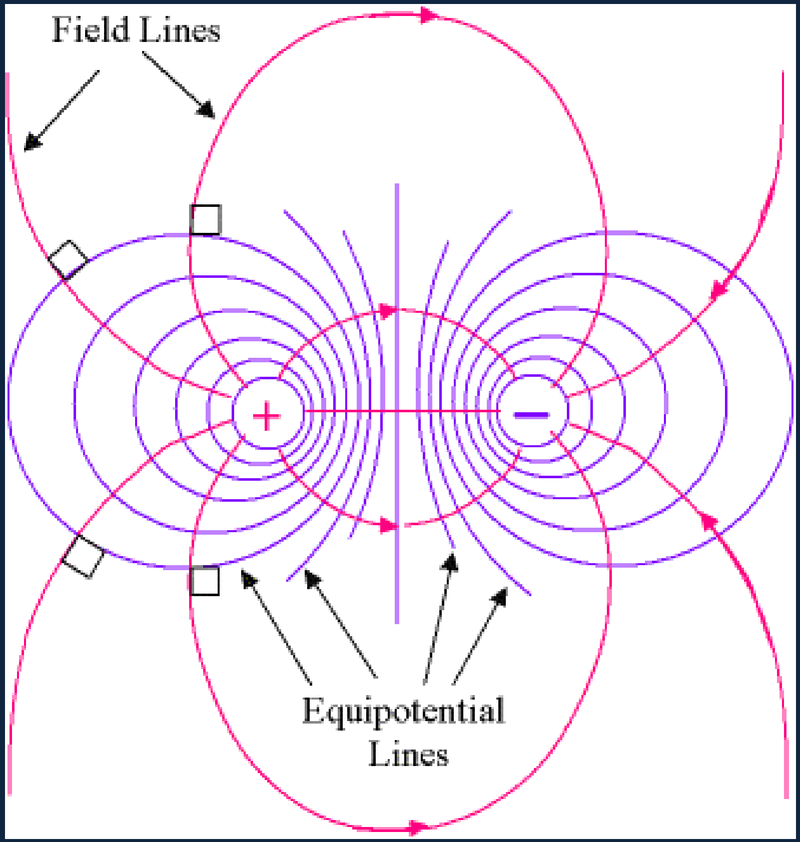
$$V(\vec{r}) = \frac{1}{4\pi\varepsilon_0}\frac{Q}{r}$$

- Obviously, equipotentials are surfaces of constant r: that is, spheres centered at the charge.
- In fact, this is also true for gravitation—the map contour lines represent where these spheres meet the Earth's surface.

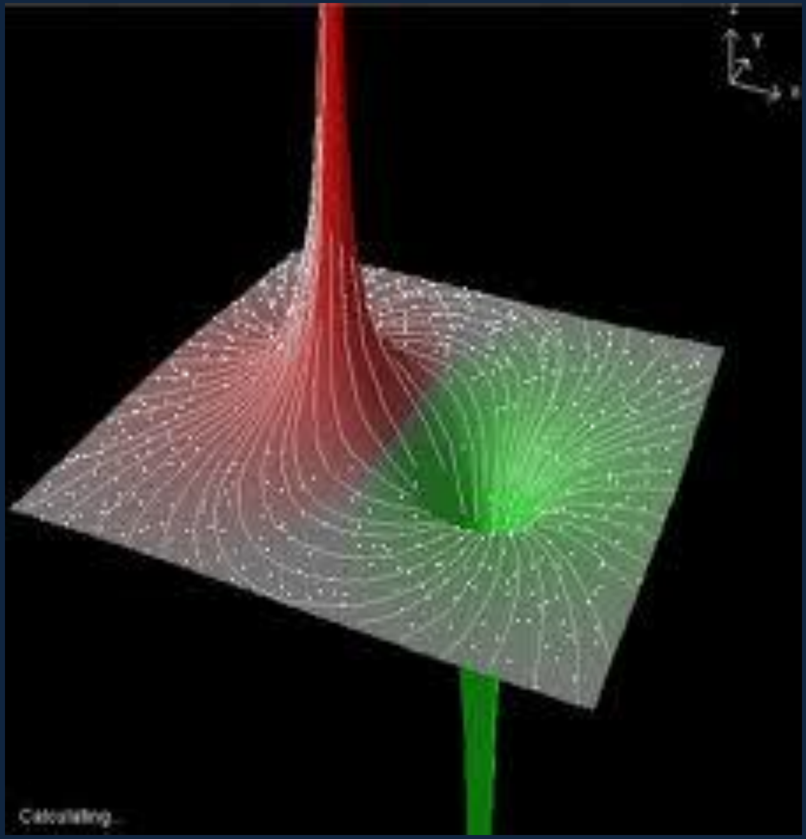
Plotting Equipotentials

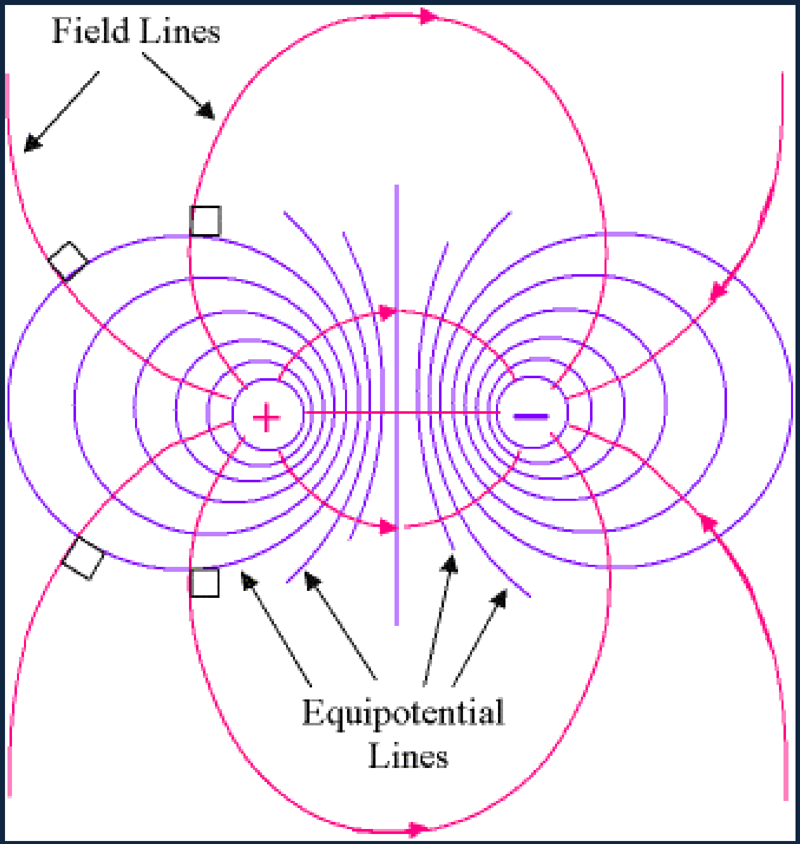
- Equipotentials are surfaces in three dimensional space—we can't draw them very well. We have to settle for a two dimensional slice.

- Check out the representations here.



Plotting Equipotentials





Here's a more physical representation of the electric potential as a function of position described by the equipotentials on the right.

Given the Potential, What's the Field?

- Suppose we're told that some static charge distribution gives rise to an electric field corresponding to a given potential $V(x, y, z)$.
- How do we find $\vec{E}(x, y, z)$?
- We do it one component at a time: for us to push a unit charge from (x, y, z) to $(x+\Delta x, y, z)$ takes work $-E_x \Delta x$, and increases the PE of the charge by $V(x+\Delta x, y, z)-V(x, y, z)$.
- So:

$$E_x = -\frac{V(x+\Delta x, y, z)-V(x, y, z)}{\Delta x} = -\frac{\partial V(x, y, z)}{\partial x} \text{ for } \Delta x \to 0.$$

What's a *Partial* Derivative?

- The derivative of $f(x)$ measures how much f changes in response to a small change in x.
- It is just the ratio $\Delta f/\Delta x$, taken in the limit of small Δx, and written df/dx.
- The potential function $V(x, y, z)$ is a function of three variables—if we change x by a small amount, <u>keeping y and z constant</u>, that's partial differentiation, and that measures the field component in the x direction:

$$E_x = -\frac{\partial V(x,y,z)}{\partial x}, E_y = -\frac{\partial V(x,y,z)}{\partial y}, E_z = -\frac{\partial V(x,y,z)}{\partial z}.$$

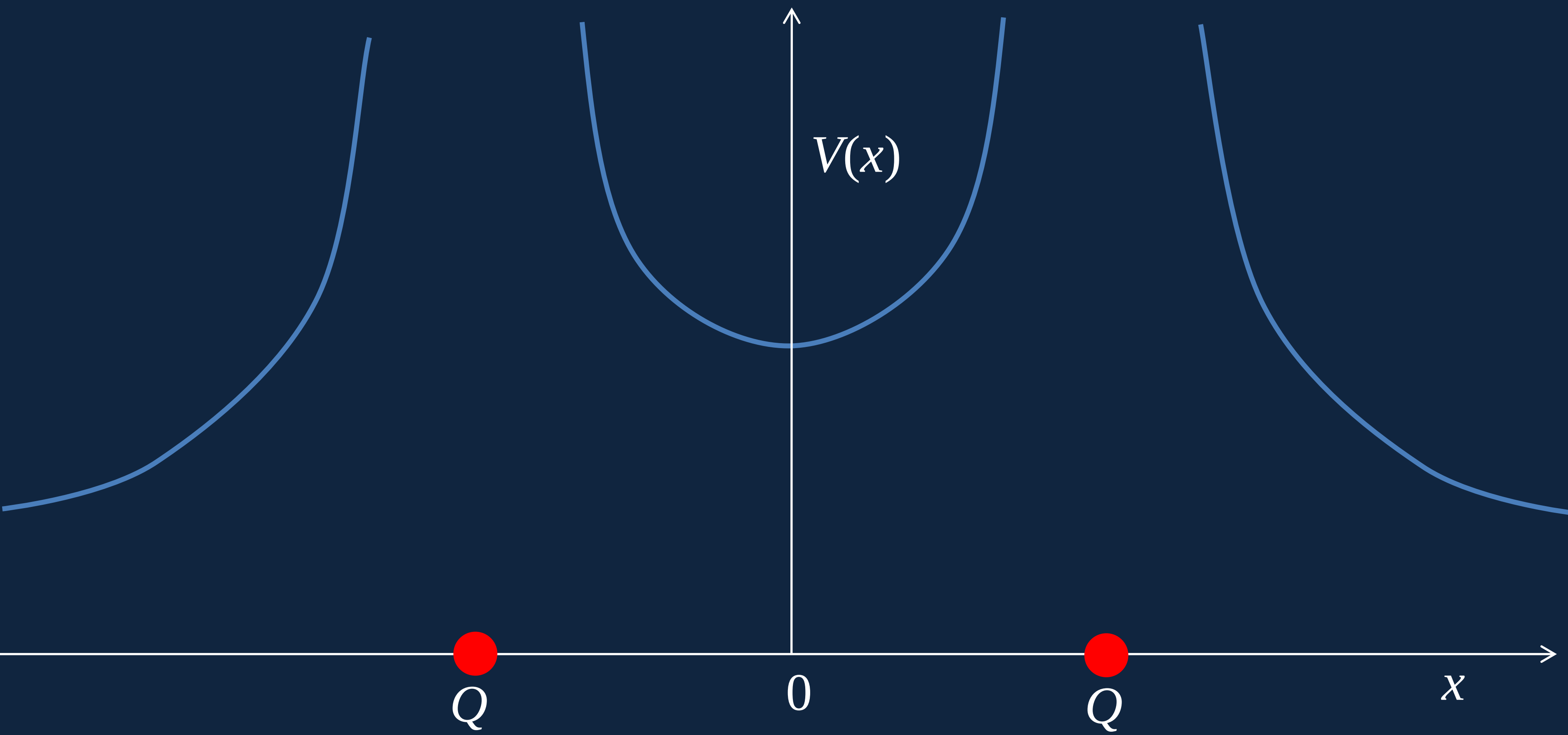
Field Lines and Equipotentials

- The work needed to move unit charge a tiny distance $\overrightarrow{d\ell}$ at position $\vec{r}$ is $-\vec{E}(\vec{r})\cdot\overrightarrow{d\ell}$.
- That is,

$$V\left(\vec{r}+\overrightarrow{d\ell}\right)-V(\vec{r})=-\vec{E}(\vec{r})\cdot\overrightarrow{d\ell}$$

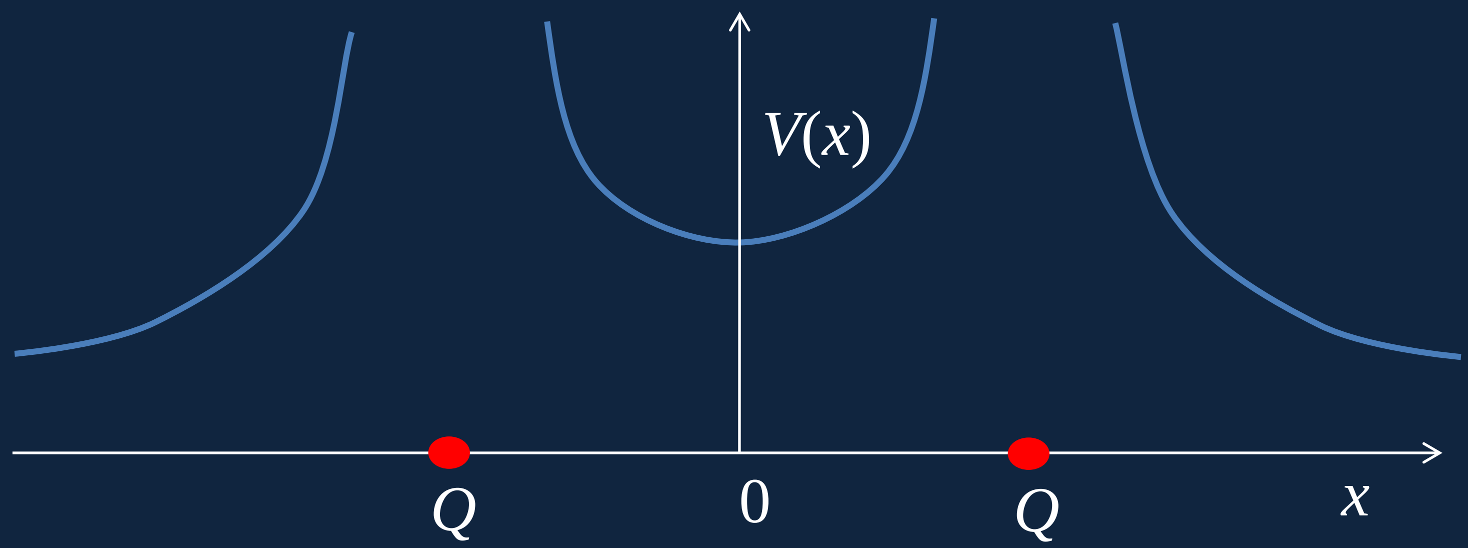
- Now, if $\overrightarrow{d\ell}$ is pointing along an equipotential, by definition V doesn't change at all!
- Therefore, the electric field vector $\vec{E}(\vec{r})$ at any point is always <u>perpendicular</u> to the equipotential surface.

Potential along Line of Centers of Two Equal Positive Charges



Note: the origin (at the midpoint) is a "saddle point" in a 2D graph of the potential: a high pass between two hills. It slopes downwards on going away from the origin in the *y* or *z* directions.

Potential along Line of Centers of Two Equal Positive Charges



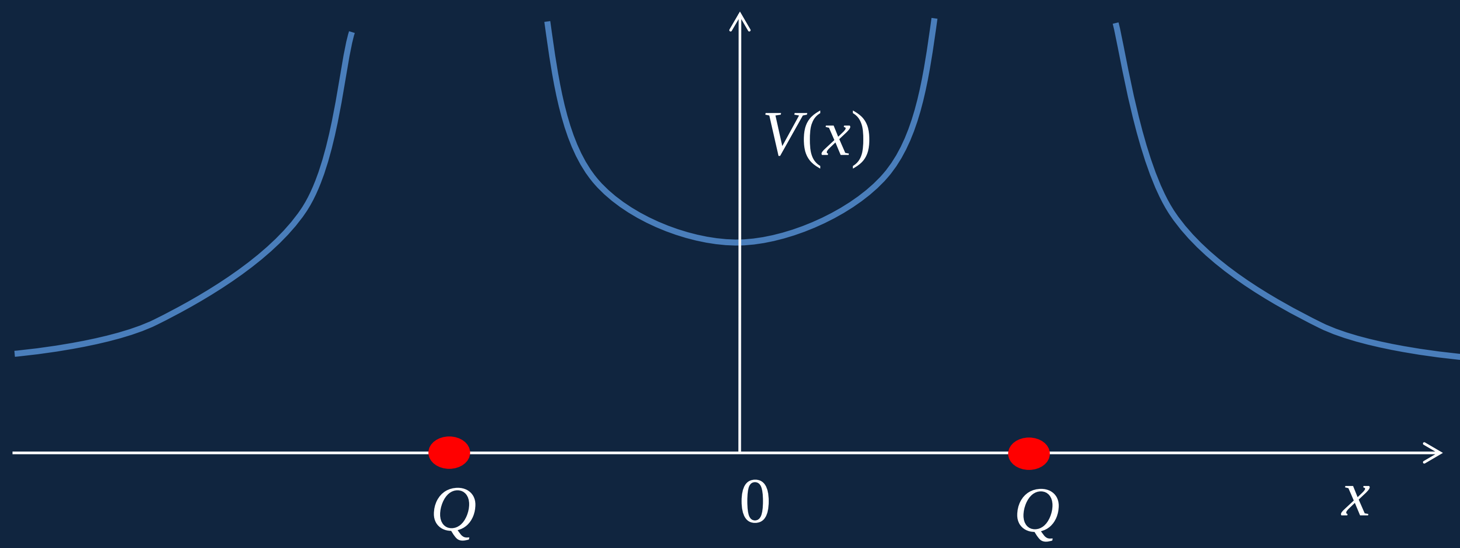
- Clicker Question:
- At the origin in the graph, the electric field E_x is:

A. maximum (on the line between the charges)

B. minimum (on the line between the charges)

C. zero

Potential along Line of Centers of Two Equal Positive Charges

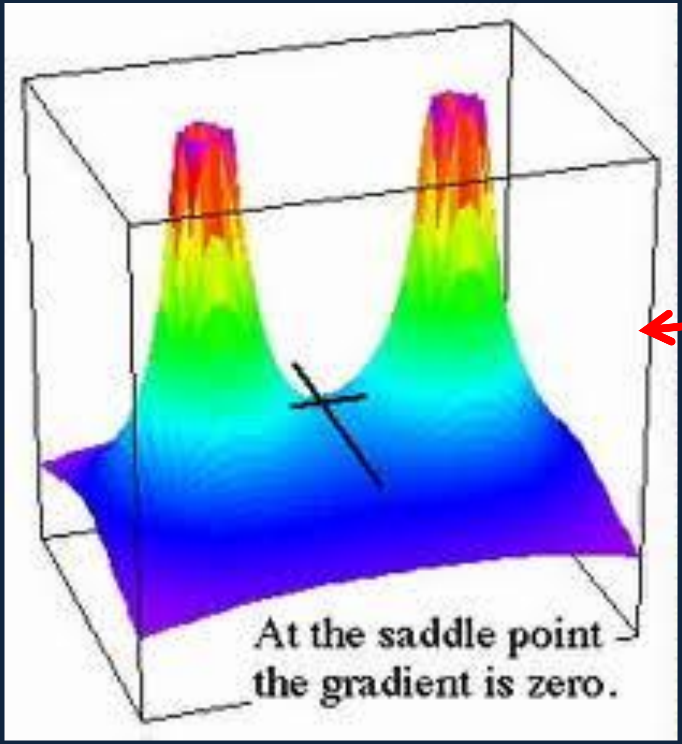


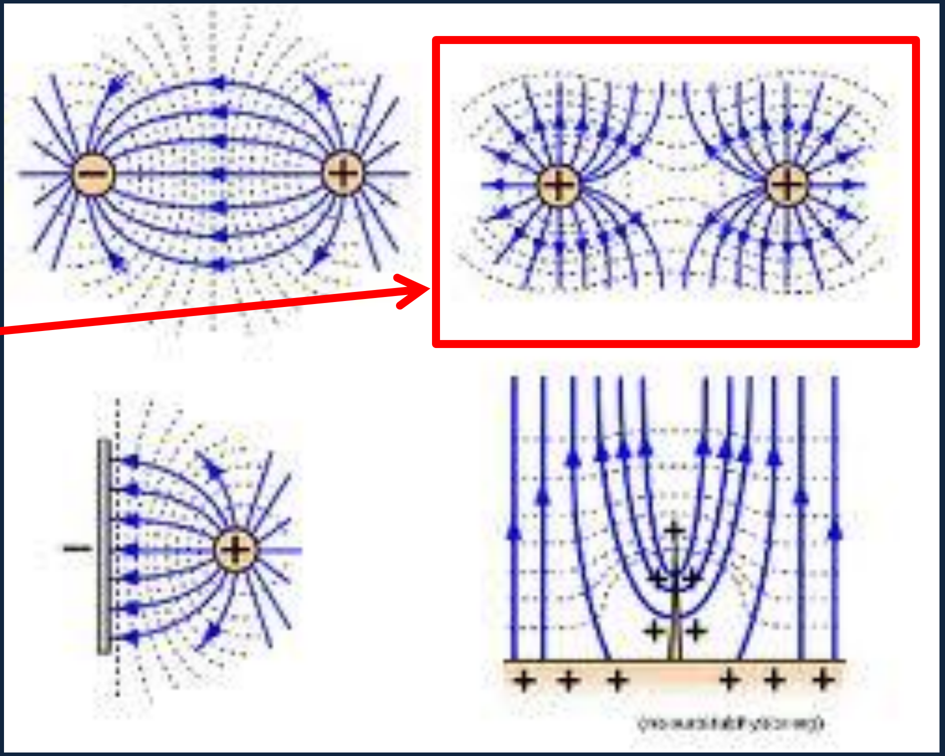
- Clicker Answer:

$E_x(0)$ = Zero: because $E_x = -\dfrac{\partial V}{\partial x}$ equals minus the slope.

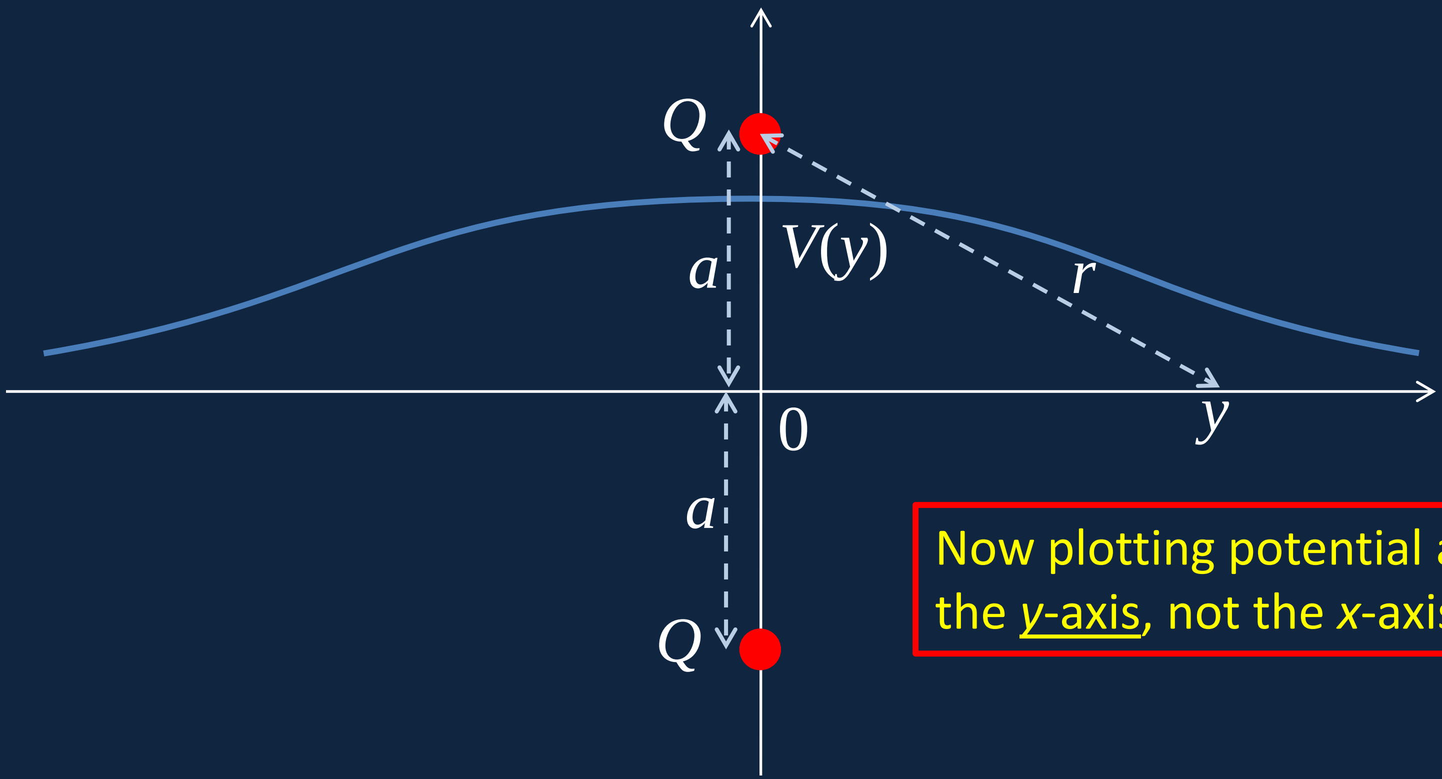
- (And of course the two charges exert equal and opposite repulsive forces on a test charge at that point.)

Potential and field from equal +ve charges





Potential along Bisector Line of Two Equal Positive Charges



Now plotting potential along the *y*-axis, not the *x*-axis!

- For charges Q at $y = 0$, $x = a$ and $x = -a$, the potential at a point on the y-axis:

$$V\left(y\right)=\frac{2kQ}{r}=\frac{2kQ}{\sqrt{a^2+y^2}}$$

Note: same formula will work on axis for a ring of charge, $2Q$ becomes total charge, a radius.

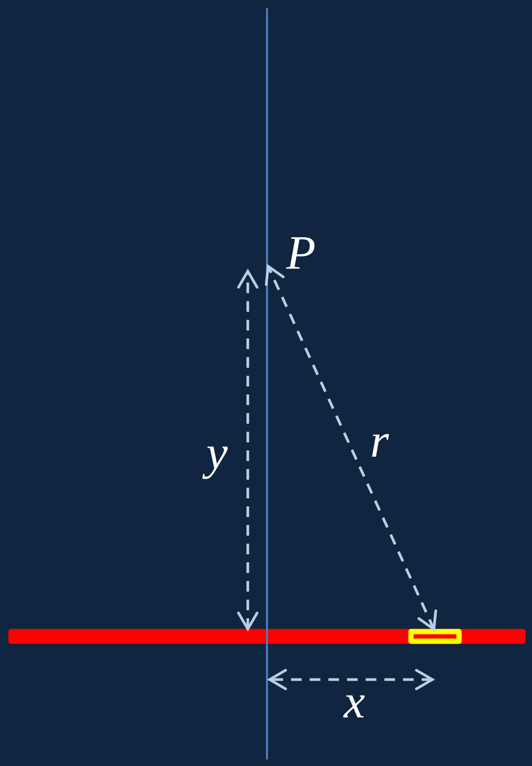
Potential from a short line of charge

- Rod of length 2ℓ has uniform charge density λ, $2\ell\lambda = Q$. What is the potential at a point P in the bisector plane?
- The potential at y from the charge between x, $x + \Delta x$ is

$$\frac{kQ_{\Delta x}}{r} = \frac{k\lambda\Delta x}{r} = \frac{k\lambda\Delta x}{\sqrt{x^2+y^2}}$$

- So the total potential

$$V(y) = \int_{-\ell}^{\ell} \frac{k\lambda dx}{\sqrt{x^2+y^2}} = \frac{kQ}{2\ell}\ln\frac{\sqrt{\ell^2+y^2}+\ell}{\sqrt{\ell^2+y^2}-\ell}$$



Great – but what does $V(y)$ look like?

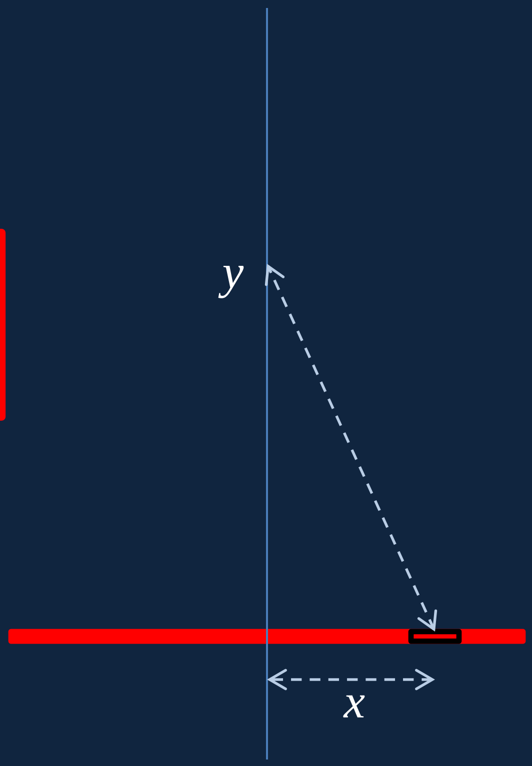
Potential from a short line of charge

$$V(y)=\int_{-\ell}^{\ell}\frac{k\lambda dx}{\sqrt{x^2+y^2}}=\frac{kQ}{2\ell}\ln\frac{\sqrt{\ell^2+y^2}+\ell}{\sqrt{\ell^2+y^2}-\ell}$$

- What does this look like at a large distance $y \gg \ell$?
- Useful math approximations: for small x, $(1+x)^{-1} \cong 1-x, \quad \ln(1+x) \cong x$
- So

$$\frac{\sqrt{\ell^2+y^2}+\ell}{\sqrt{\ell^2+y^2}-\ell} \cong \frac{y+\ell}{y-\ell}=\frac{1+(\ell/y)}{1-(\ell/y)} \cong 1+2(\ell/y)$$

- And $V(y)=\dfrac{kQ}{2\ell}\ln(1+2\ell/y) \cong \dfrac{kQ}{y}$

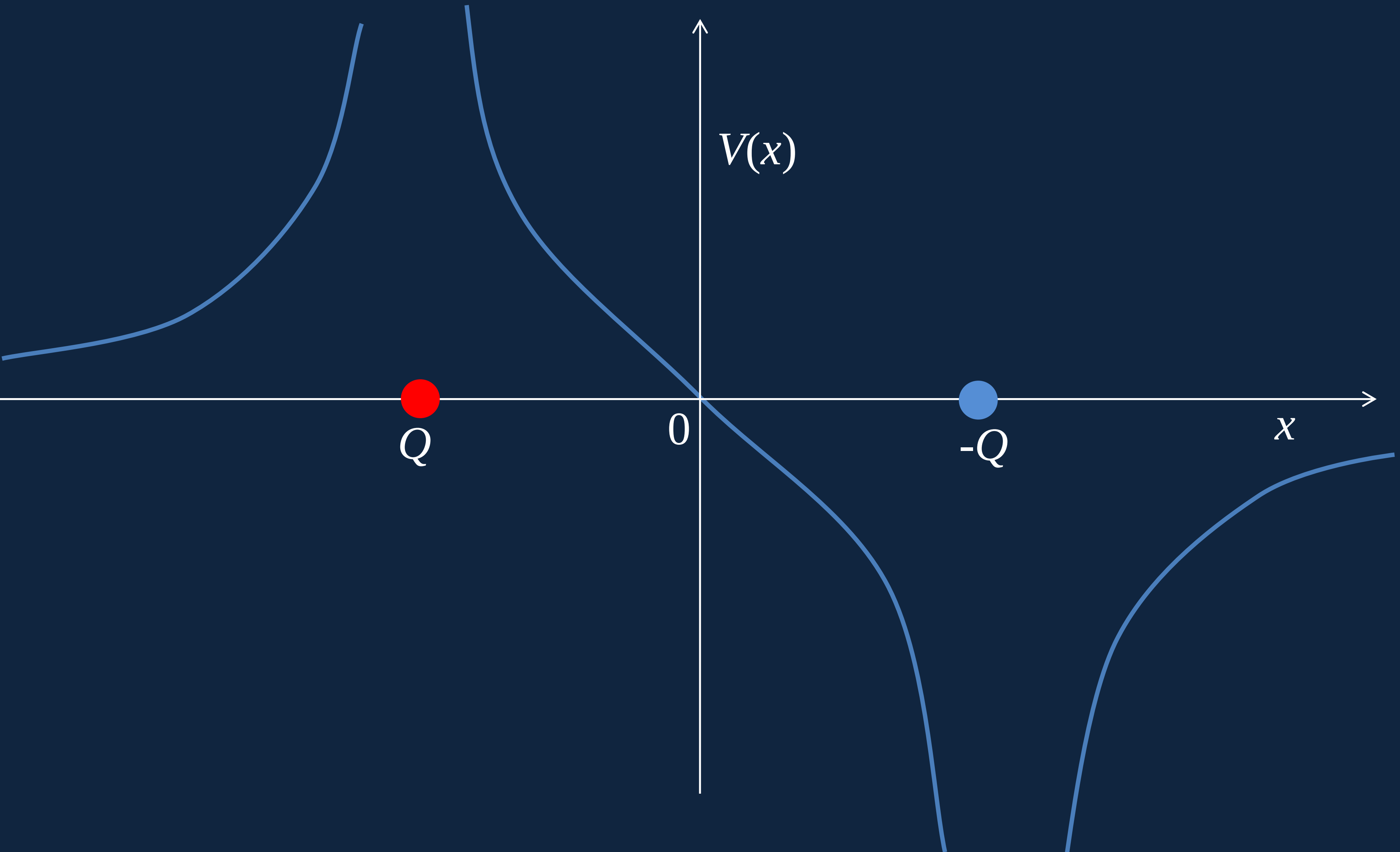


Bottom line: at distances large compared with the size of the line, it looks like a point charge.

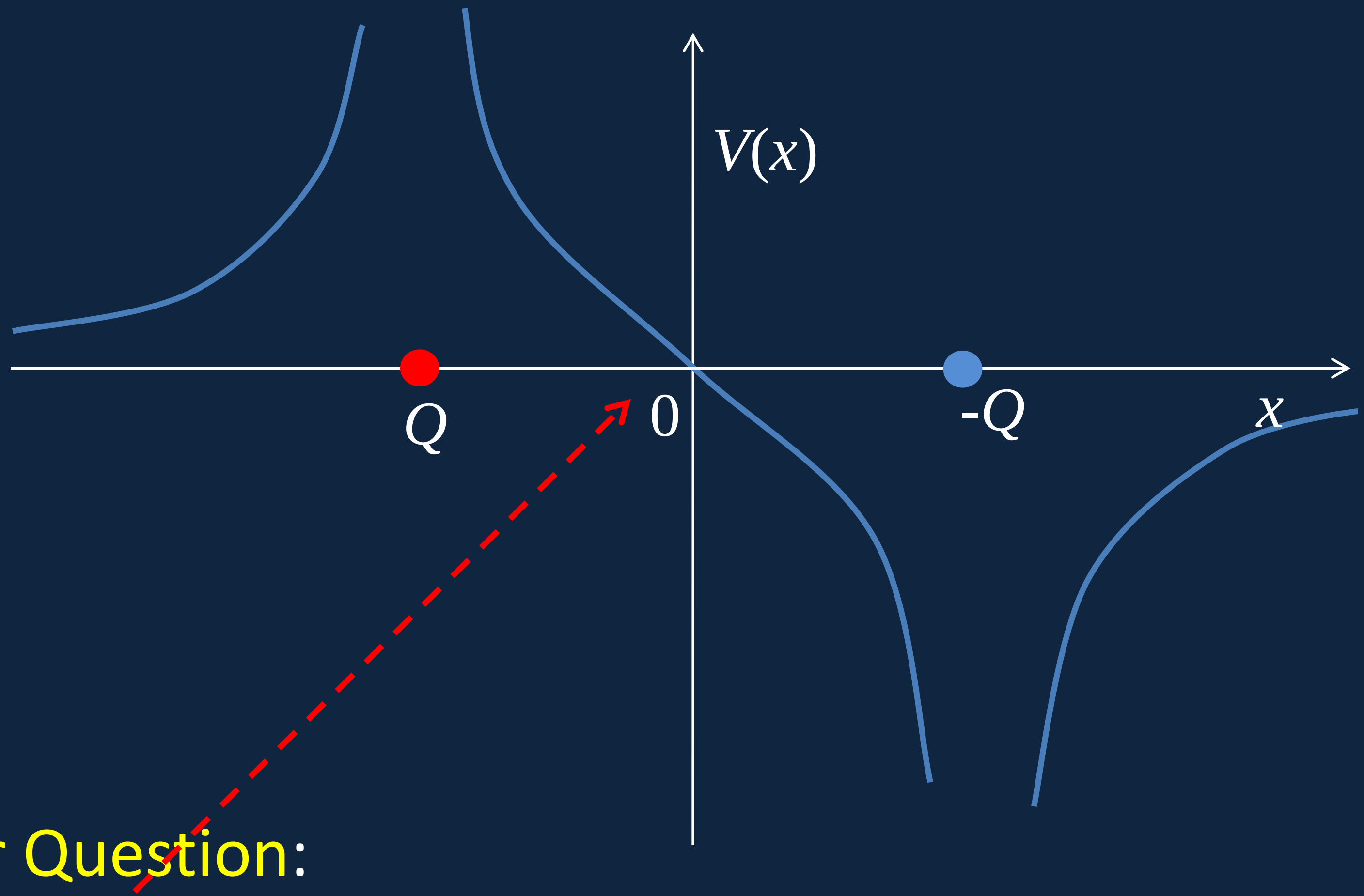
Potential from a *long* line of charge

- Let's take a conducting cylinder, radius *R*.
- If the charge per unit length of cylinder is λ, the external electric field points radially outwards, from symmetry, and has magnitude $E(r) = 2k\lambda/r$, from Gauss's theorem.
- So $$V(r) = V(R) - \int_R^r \vec{E}(\vec{r}\,')\cdot d\vec{r}\,' = V(R) - 2k\lambda \int_R^r \frac{dr'}{r'}$$
$$= V(R) - 2k\lambda(\ln r - \ln R).$$
- Notice that for an infinitely long wire, the potential keeps on increasing with *r* for ever: we can't set it to zero at infinity!

Potential along Line of Centers of Two Equal but Opposite Charges



Potential along Line of Centers of Two Equal but Opposite Charges



Clicker Question:

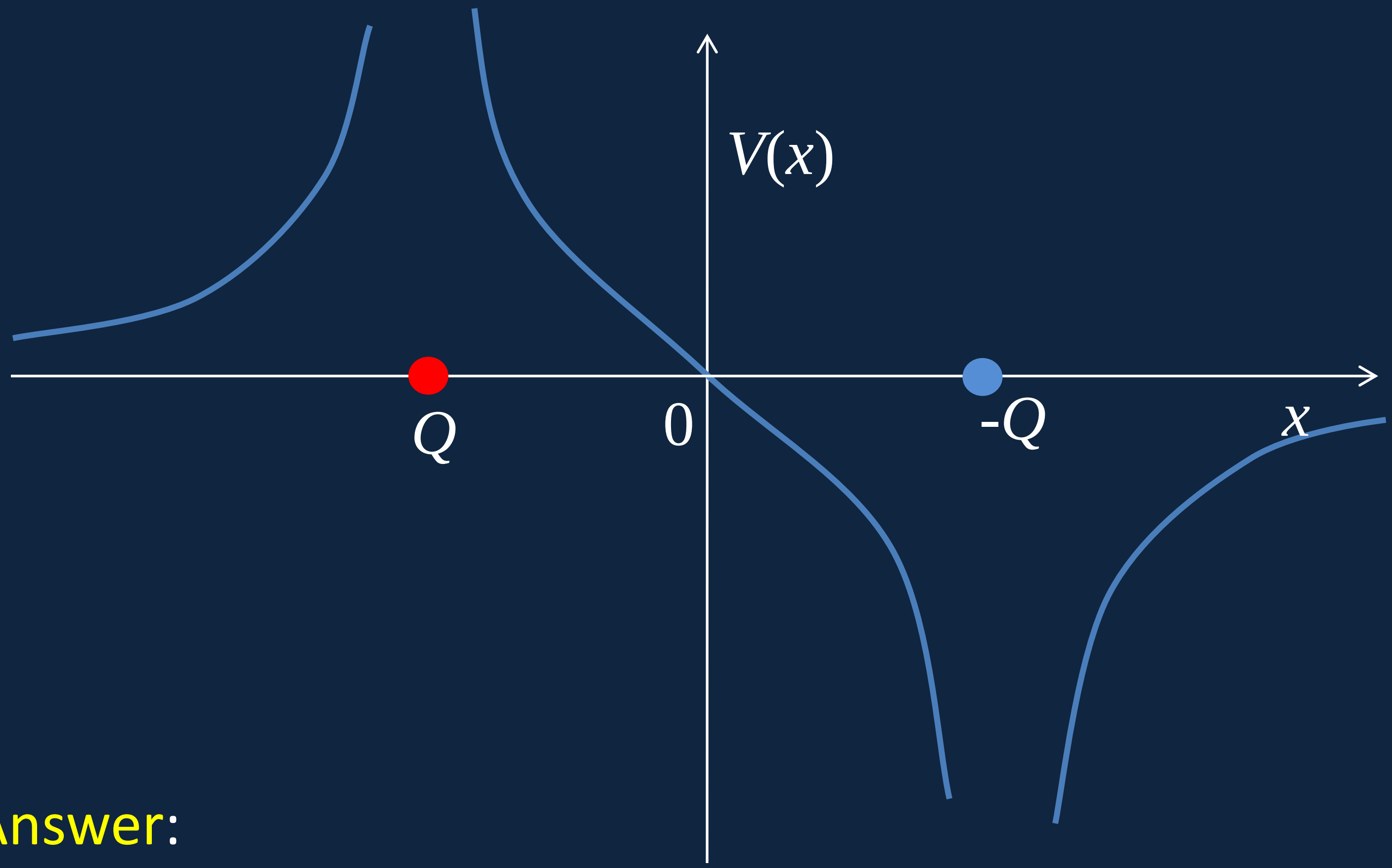
At the origin, the electric field magnitude is:

A. maximum (on the line and *between* the charges)

B. minimum (on the line and *between* the charges)

C. zero

Potential along Line of Centers of Two Equal but Opposite Charges



Clicker Answer:

At the origin in the above graph, the electric field magnitude is: minimum (on the line between the charges) ←

- Remember the field strength is the slope of the graph of $V(x)$: and between the charges the slope is least steep at the midpoint.

Charged Sphere Potential and Field

- For a spherical conductor of radius R with total charge Q uniformly distributed over its surface, we know that

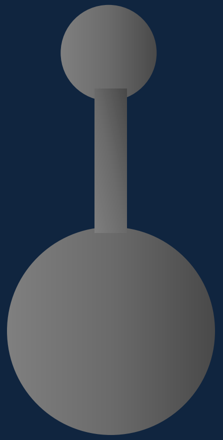
$$\vec{E}(\vec{r}) = \frac{1}{4\pi\varepsilon_0}\frac{Q\hat{r}}{r^2} \quad \text{and} \quad V(\vec{r}) = \frac{1}{4\pi\varepsilon_0}\frac{Q}{r}.$$

- The field at the surface is related to the surface charge density σ by $E = \sigma/\varepsilon_0$.
- Note this checks with $Q = 4\pi R^2 \sigma$.

Connected Spherical Conductors

- Two spherical conductors are connected by a conducting rod, then charged—all will be at the same potential.
- Where is the electric field strongest?

A. At the surface of the small sphere
B. At the surface of the large sphere
C. It's the same at the two surfaces.

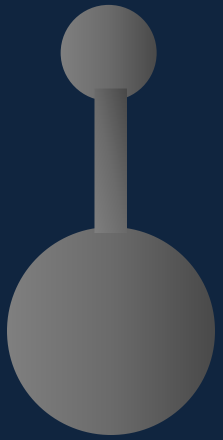


Connected Spherical Conductors

- Two spherical conductors are connected by a conducting rod, then charged—all will be at the same potential.
- Where is the electric field strongest?

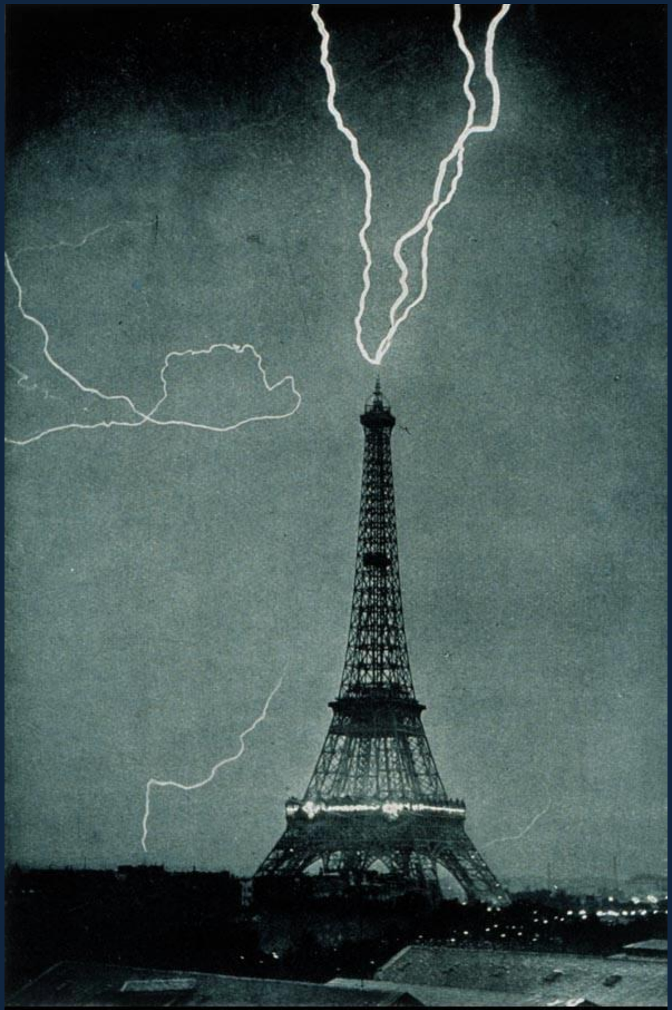
A. <u>At the surface of the small sphere</u>.

- Take the big sphere to have radius R_1 and charge Q_1, the small R_2 and Q_2.
- Equal potentials means $Q_1/R_1 = Q_2/R_2$.
- Since $R_1 > R_2$, field $kQ_1/R_1^2 < kQ_2/R_2^2$.
- This means the surface charge density is greater on the smaller sphere!



Electric Breakdown of Air

- Air contains free electrons, from molecules ionized by cosmic rays or natural radioactivity.
- In a strong electric field, these electrons will accelerate, then collide with molecules. If they pick up enough KE between collisions to ionize a molecule, there is a "chain reaction" with rapid current buildup.
- This happens for E about 3×10^6 V/m.



Photograph of lightning striking the Eiffel Tower, June 3, 1902, taken by M.G. Loppé.

Voltage Needed for Electric Breakdown

- Suppose we have a sphere of radius 10cm, 0.1m.
- If the field at its surface is just sufficient for breakdown,

$$3\times10^{6} = \frac{1}{4\pi\varepsilon_0}\frac{Q}{R^2}$$

- The voltage

$$V = \frac{1}{4\pi\varepsilon_0}\frac{Q}{R} = 3\times10^{6} R = 300,000V$$

- For a sphere of radius 1mm, 3,000V is enough—there is discharge before much charge builds up.
- This is why lightning conductors are pointed!