Draft Notes on Momentum

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Center of Mass

Let's begin with two objects, say two children on a seesaw. You already know that if one child weighs twice as much as the other, the seesaw will balance if the lighter child is twice as far from the center, the pivot point of the seesaw. The **center of mass** of the two children is the balance point—the center of the seesaw when it's balanced. It is **not**, as you might at first think, a point with half the mass on one side and half on the other, but the point about which the **leverage** of one mass, equals that of the other mass (in the opposite direction). It's also referred to as the center of gravity. It plays a big role in dynamics because, as we shall see, if several interacting bodies are moving around, the center of mass moves in a particularly simple way, and that can be a key to a better understanding of what's going on.

To be a little more formal, if we have two point masses m_1, m_2, on the x-axis at x_1, x_2 respectively, then the center of mass x_{cm} is at the balance point, that is,

$$m_1(x_{\text{cm}} - x_1) = m_2(x_2 - x_{\text{cm}})$$

so

$$x_{\text{cm}} = \frac{m_1x_1 + m_2x_2}{m_1 + m_2}$$

If we now add a third child to the seesaw and rebalanced, then taking the pivot point as the x-axis origin, the total leverage is zero

$$m_1x_1 + m_2x_2 + m_3x_3 = 0$$

and if we don't take it as the origin,

$$m_1(x_1 - x_{\text{cm}}) + m_2(x_2 - x_{\text{cm}}) + m_3(x_3 - x_{\text{cm}}) = 0$$

from which

$$x_{\text{cm}} = \frac{m_1x_1 + m_2x_2 + m_3x_3}{m_1 + m_2 + m_3}$$

Evidently this expression can be generalized to any number of masses, so

$$x_{\text{cm}} = \frac{\sum_{i=1}^{N} m_i x_i}{\sum_{i=1}^{N} m_i}$$

Actually finding the center of mass of a system **can be done in stages**.

Take the case of the three masses, for example. Let's write the center of mass of m_1, m_2 as x_{12}, and total mass $m_1 + m_2 = m_{12}$. Then the center of mass of m_1, m_2 and m_3 is the same as the center of mass of m_3 at x_3 and m_{12} at x_{12}:

$$\begin{aligned}\frac{m_3x_3 + m_{12}x_{12}}{m_3 + m_{12}} &= \frac{m_3x_3 + (m_1 + m_2)(m_1x_1 + m_2x_2)/(m_1 + m_2)}{m_3 + m_1 + m_2} \\ &= \frac{m_1x_1 + m_2x_2 + m_3x_3}{m_1 + m_2 + m_3}.\end{aligned}$$

In fact, the formula for center of mass can be extended to an infinite number of masses, for example a continuous distribution of mass along the line: the sum will then become an integral.

For example, suppose that a straight wire of variable thickness lies along the x-axis from 0 to L, and it has mass $\lambda(x)dx$ in the interval between x and $x + dx$. Its total mass is given by summing over all the intervals, that is,

$$M = \int_0^L \lambda(x)dx$$

and its center of mass, generalizing the formula for finite numbers of masses above, is

$$x_{\text{cm}} = \frac{\int_0^L \lambda(x)x dx}{M}.$$

Center of Mass in Two or Three Dimensions

The simplest possible case is to take three equal masses, let's take them all equal to m kilograms. We denote their positions by $\vec{r}_1, \vec{r}_2, \vec{r}_3$. Since $\vec{r}_2, \vec{r}_3$ lie on a line, we already know how to find their center of mass: it's just the midpoint, $(\vec{r}_2 + \vec{r}_3)/2$, and their combined mass $M = 2m$.

The next step is to find the center of mass of a mass $2m$ at $(\vec{r}_2 + \vec{r}_3)/2$ and a mass m at $\vec{r}_1$. This is again two masses in a line (of course!) so the result is:

$$\vec{r}_{\text{cm}} = \frac{m\vec{r}_1 + 2m(\vec{r}_2 + \vec{r}_3)/2}{3m} = \frac{m\vec{r}_1 + m\vec{r}_2 + m\vec{r}_3}{3m} = \frac{\vec{r}_1 + \vec{r}_2 + \vec{r}_3}{3},$$

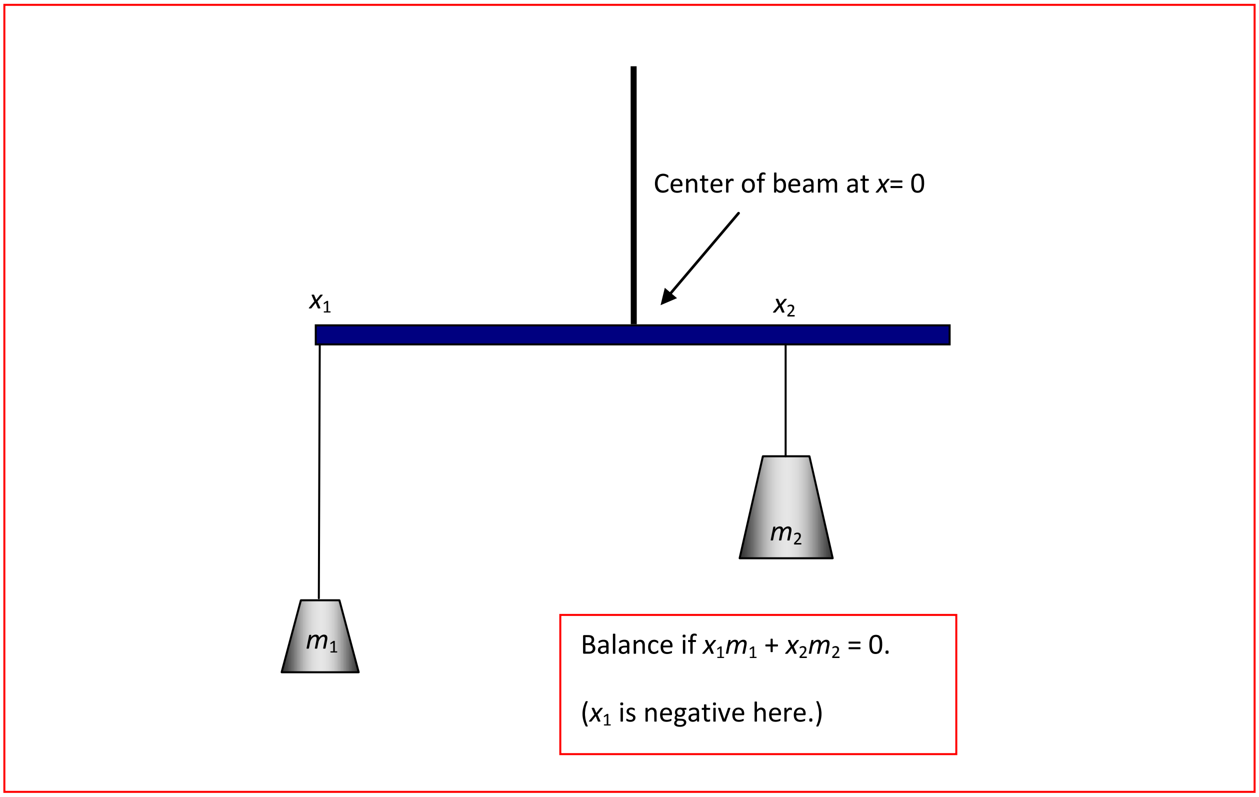
a nicely symmetrical result: obviously it didn't matter which two masses we chose to begin with.

The same analysis works for unequal masses, to give

$$\vec{r}_{\mathrm{cm}} = \frac{m_1\vec{r}_1 + m_2\vec{r}_2 + m_3\vec{r}_3}{m_1 + m_2 + m_3}$$

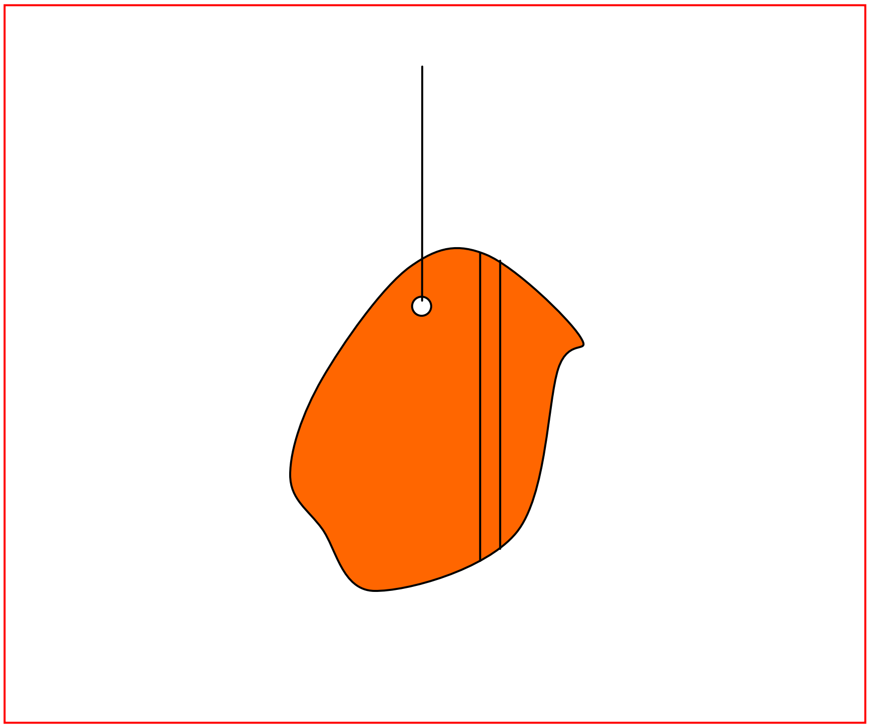
and the generalization to more masses, and to a continuous mass distribution, is exactly as in one dimension, except that x is replaced by $\vec{r}$.

Now we're in more than one dimension, we can generalize the two children on a seesaw as follows: we can imagine two weights hung from a uniform beam, which is itself hung on a string:



The point is that the vertical positioning of the weights, the y-coordinate, does not affect the question of balance (taking the string mass negligible): all that counts is the x-coordinate.

It follows that for a collection of masses, or an irregular shape, hung from a string under gravity will come to rest with the center of mass (or gravity) directly below the point of suspension:



To see this, note that when it is freely hanging, suspended from a string at $x = 0$, we can imagine it divided into vertical strips, like that between x and $x + dx$, which will have mass $m(x)dx$, say, and leverage about the line of suspension of $xm(x)dx$. This means that the center of mass is somewhere on this line. To find out where, we need to suspend the shape from some other point, find the vertical line through the new point of suspension, and find where they intersect.

Significance of the Center of Mass in Dynamics

Suppose we have a system of N masses, interacting with each other and also subject to external forces. If the total mass of the system is M,

$$M\vec{r}_{\text{cm}} = \sum_{i=1}^{N} m_i \vec{r}_i$$

and differentiating,

$$M\vec{v}_{\text{cm}} = \sum_{i=1}^{N} m_i \vec{v}_i,$$

meaning the **total momentum of the system is total mass × velocity of the center of mass**.

Now, the rate of change of momentum for each mass is equal to the force acting on that mass. But internal forces—forces between masses in the system—change the momenta of the pair of particles involved by equal and opposite amounts, so cannot affect the total momentum of the whole system.

This means the rate of change of the total momentum of the system equals the sum of the **external** forces acting on it.

In particular, **if the system initially has zero total momentum, and no external forces act, the center of mass will remain at rest**: if you walk from one end of a initially stationary boat to the other end, the center of mass of you plus the boat will not move, neglecting as usual tiny frictional effects from the water.

The Center of Mass Frame of Reference

Galileo was the first to spell out that the laws of physics are the same in a frame of reference moving at a steady velocity as in one at rest. Of course, Newton's laws hadn't been invented at that point, the way he put it was that if you're inside a big ship moving steadily, you can't tell you're moving by observing a ball you throw across the room, or liquid dripping from a bottle, or various other things—his point was that the Earth was probably moving, and you wouldn't be able to tell.

This invariance of the laws of physics is called **Galilean invariance**, and in fact generalizing this idea to include light led Einstein to the Theory of Relativity.

It's often helpful in analyzing collisions to look in the frame of reference in which the center of mass is at rest. That's because if there are no external forces, just the forces between the colliding objects, the center of mass remains at rest in that frame.

For example, consider the case of two masses in one dimension undergoing elastic collision. For visualization purposes, let's suppose there is a light spring between them. If the two masses are m_1, m_2, then in the center of mass frame they approach each other from opposite directions with velocities in inverse ratios to their masses—remember the total momentum is zero in the center of mass frame.

Now, during the actual collision, when the spring is maximally compressed the masses are not moving relative to each other (otherwise the spring length would be changing). But this means that in the center of mass frame, they're not moving at all. Therefore, we can easily read off how much energy is stored in the spring at this point, it's equal to the kinetic energy of the two masses in the center of mass frame before they collided.

It's also worth reviewing how they separate in this frame: as the spring decompresses, it obviously goes through the identical force pattern on the masses that occurred on compression. Therefore, each of the masses will be accelerated by the spring by exactly the same amount it was decelerated during compression. **This means that in the center of mass frame the masses will emerge from the collision with their velocities the exact reverse of those they went in with.**

Collisions in Two Dimensions

As in one dimension, it's worth thinking in the center of mass frame, but this time the velocities are not reversed in general. This becomes evident on considering two pool balls colliding—they come off at different angles depending on how they struck each other, head on, a glancing blow, or in between. You should play around with the Collision Applet here.

A general point in analyzing these collisions is that it sometimes pays to work with vectors rather than immediately writing out the equations for components.

For example, consider the case of a particle traveling at $\vec{v}$ hitting an identical particle at rest, and they fly apart at $\vec{v}_1, \vec{v}_2$. The equations for momentum and energy conservation are

$$\vec{v} = \vec{v}_1 + \vec{v}_2, \quad v^2 = v_1^2 + v_2^2$$

and it follows immediately from Pythagoras' theorem that the two emerging velocities are at right angles to each other.

If the masses are not equal, things are more complicated, but it's still a good idea not to go to components right away. Consider the case of the second particle having twice the mass:

$$\vec{v} = \vec{v}_1 + 2\vec{v}_2, \quad v^2 = v_1^2 + 2v_2^2.$$

Squaring the first equation gives $v^2 = \left(\vec{v}_1 + 2\vec{v}_2\right)^2 = v_1^2 + 4v_2^2 + 4v_1 v_2 \cos\theta$, and equating this to the energy equation give a relation between v_1, v_2 and $\cos\theta$. You could also square $\vec{v} - \vec{v}_1 = 2\vec{v}_2$, etc.