Rotational Motion

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Preliminaries: Units for Angular Velocity

The tachometer on your car dashboard tells you your car engine's angular speed in rpm, revolutions per minute, and probably redlines around 6 or 7,000 rpm.

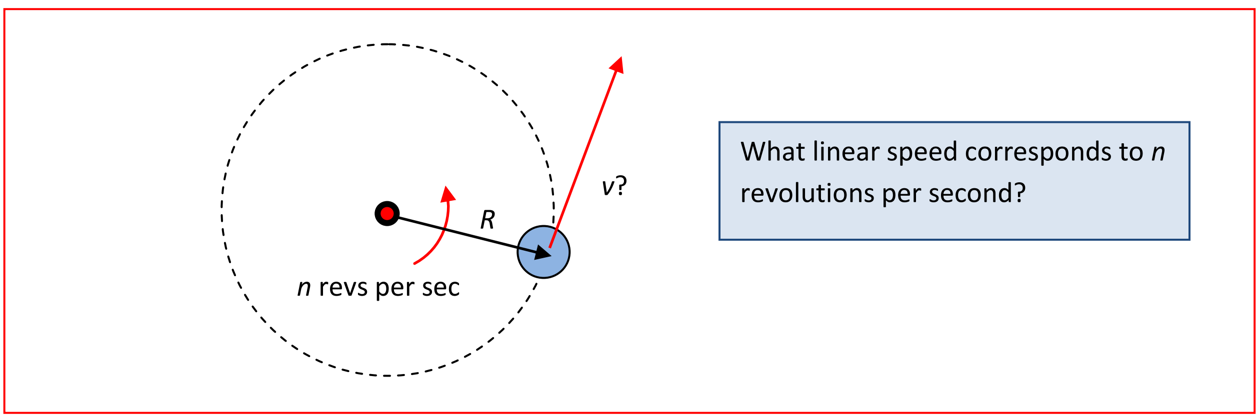
A common unit for electromagnetic oscillations (to be discussed nest semester) is cycles per second: there's even an official name, the Hertz, abbreviated Hz:

1 Hz = one cycle per second

But angular motion isn't restricted to complete cycles—we need to measure fractions of a cycle as well, for example, in reorienting an aircraft carrier, the Hertz is not going to be a handy unit.

A complete circle is divided into 360 degrees, written 360°. (This was done by the Babylonians, 4,000 years ago, probably because the Sun moves very close to 1° a day through the sky, relative to the fixed stars.) So we could talk about angular velocity in degrees per second.

However, from a practical point of view, we often need to relate angular velocity to the speed of some part of the wheel. For example, if I whirl a stone around in a sling at some number of revolutions per second, or degrees per second, then let go, how fast is the stone moving?

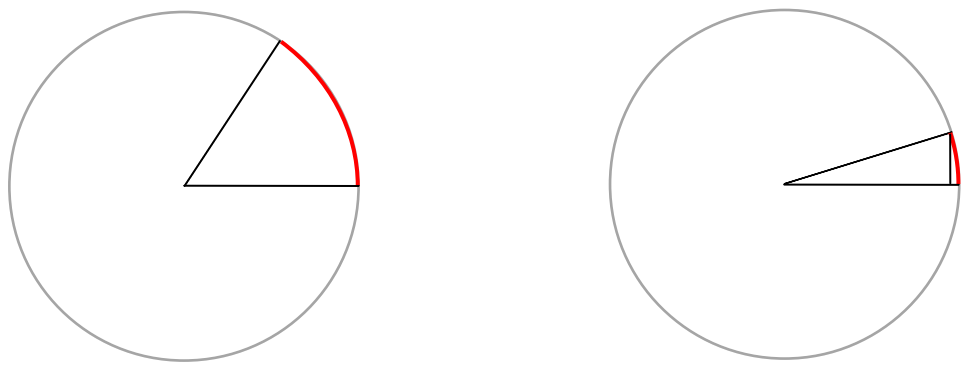


Let's say the angular velocity is n degrees per second, or $n/360$ Hz. If the sling is r meters long, so the stone is going in a circle of radius r (ignoring the slight movement of my hand), the stone travels $2\pi r$ meters per cycle, so its speed v is given in terms of its angular speed n by:

$$v = \frac{2\pi}{360} nr.$$

This rather clumsy formula is a consequence of following the Babylonians in defining the unit of angle as 1/360 of a full circle. We'd have a much nicer formula if instead we defined our unit of angle as $1/2\pi$ of a full circle!

So we define: **the radian = $1/2\pi$ of a full circle**. This works out to be about 57.3°, and since the complete circumference of a circle is $2\pi r$, the distance around the circle corresponding to an angle of one radian is one radius. Duh.



Angle of **one radian**

the red solid arc has length equal to the radius of the circle.

$\sin\theta \approx \theta$ for θ small

for $r = 1$, the arc length is θ, the short black line = $\sin\theta$.

It's easy to see that radians are the natural angle unit for trig: look at the sine of a small angle.

Our default notation for an angle is ϑ, measured from the x-axis, with counterclockwise as positive.

Angular velocity in **radians per second** will usually be denoted by ω: $\omega = d\theta / dt$.

So if a wheel is rotating at positive angular velocity ω radians per second about a fixed axle, a point on the wheel at distance r from the axle is moving at speed

$$\boxed{v = r\omega}$$

a formula we'll be using a lot.

The standard notation for **angular acceleration** is $\alpha = d\omega / dt = d^2\theta / dt^2$.

Notice that if we have **constant angular acceleration**, formulas for angular velocity and angular displacement (meaning angle turned through) are just like those for linear motion under constant force:

$$\omega = \omega_0 + \alpha t$$
$$\theta = \theta_0 + \omega_0 t + \tfrac{1}{2}\alpha t^2$$
$$\omega^2 = \omega_0^2 + 2\alpha(\theta - \theta_0).$$

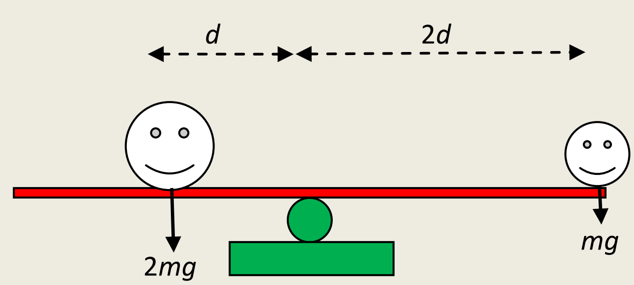
Just be sure your system has **constant** angular acceleration before using these formulas!

Torque: Seesaw Physics

Torque is a measure of the effectiveness of a force in getting something rotating: actually we've come across this already in the discussion of the seesaw. If a child sits at the end of a level empty seesaw, it begins to rotate. O.K., it doesn't get far, but there *is* angular acceleration. And we know the if the child sits close to the pivot point, the seesaw turns more slowly.

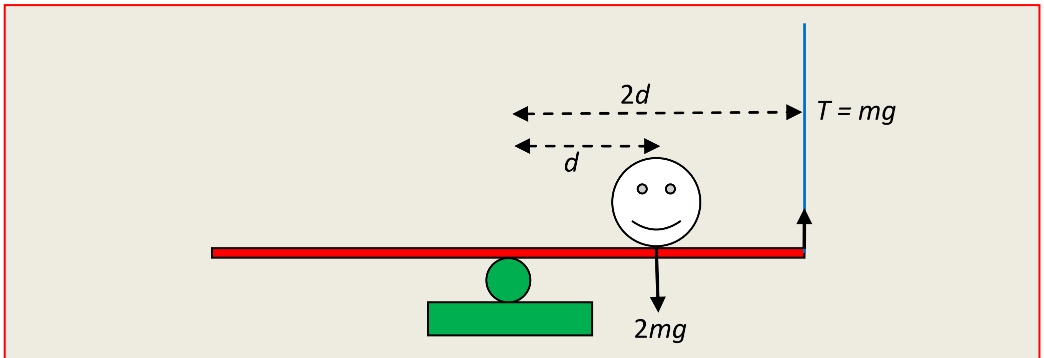
We've also seen that two children, one twice as heavy sitting at half the distance, balance—we said they had the same **leverage**. This "leverage" is actually what's called **torque**. It's also called the "moment" of a force about an axis of rotation. It's the product of the weight of the child and the distance from the pivot or axle.

So, for a 20 kg child at a distance of 1.5m, the torque about the pivot point is 300 N.m, setting g = 10.



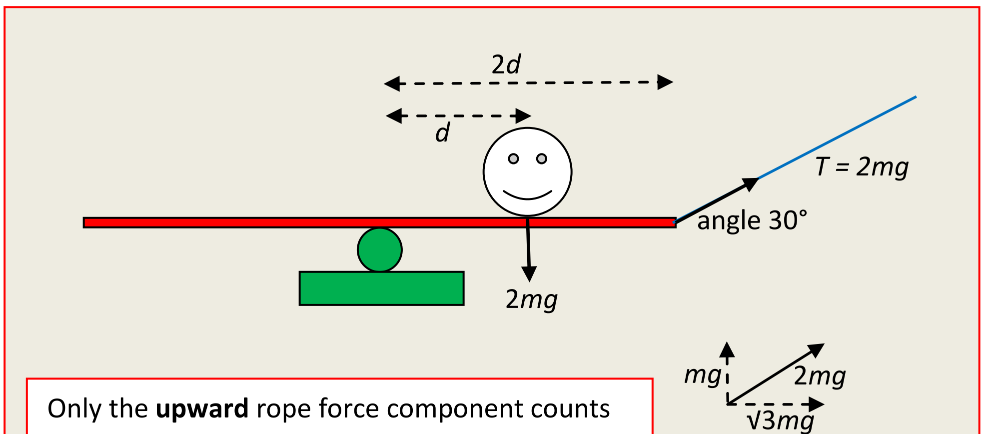
Balancing a seesaw

We can also have two forces on the same side of the pivot balancing—if one of them is upwards. If a 20kg child is sitting half way to the pivot, with no-one on the other side, you could lift the seesaw to a horizontal position by supplying an upward force equivalent to 10 kg weight at the end of the seesaw, the same side as the child.



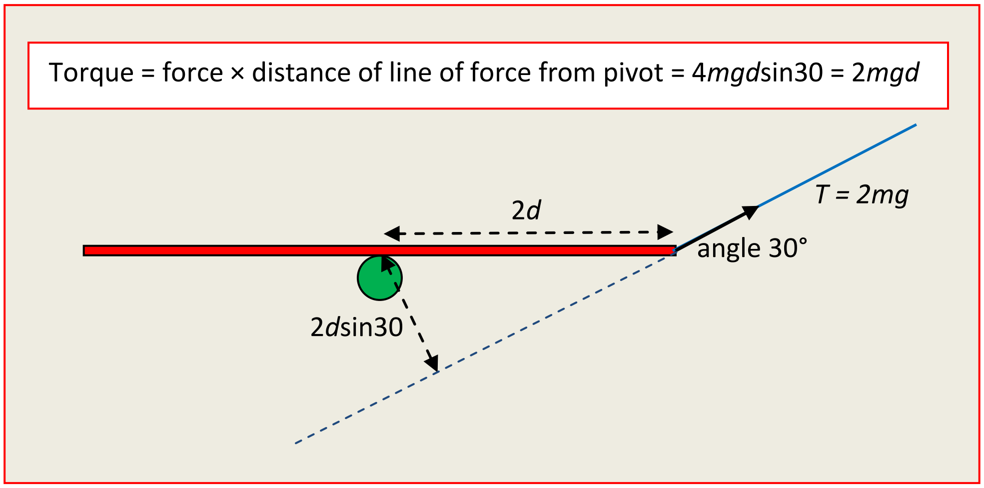
Balancing a seesaw the hard way

One further point: what if you supplied the upward force by tying a (light) rope to the end of the seesaw, and tugging at some angle? In that case, the tension in the rope, where it pulls on the seesaw, is equivalent to a force perpendicular to the seesaw plus a force parallel. The latter component does nothing (unless the seesaw can move sideways), the upward component is all that counts.



Only the **upward** rope force component counts

There's another way to look at this which is sometimes useful: instead of resolving the force into components at its point of application, as in the diagram above, we can take the whole force, acting along the line in space through its point of application, but calculate its leverage—the torque—by taking the lever arm to be the distance from the pivot point to the line of the force:



Either way, the formula for torque strength is:

$$\tau = Fr\sin\theta$$

where *F* is the magnitude of the force, *r* the distance from the pivot of the point of application, and ϑ the angle between the vector $\vec{r}$ from the pivot point to where the force acts and the force $\vec{F}$.

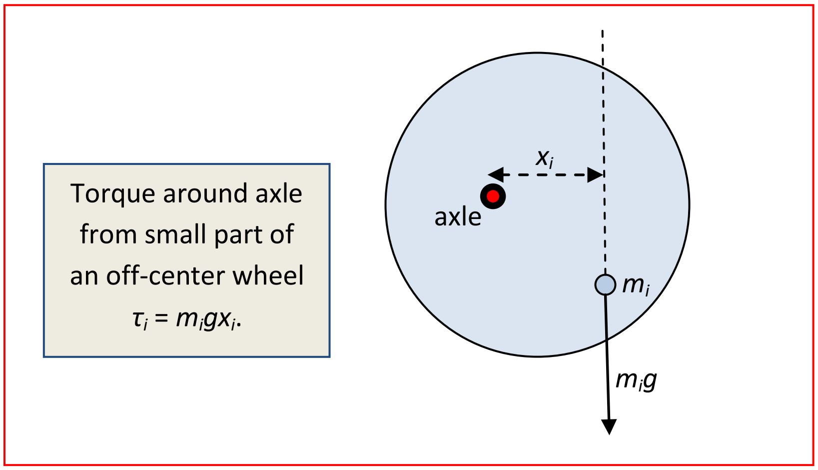
Rotational Equivalent of Newton's First Law

By the time Galileo noticed that a smooth ball on a flat surface with little friction would keep the same velocity a long time, it was already known that a rotating grindstone, with a well-oiled axle, would keep spinning a long time too—and this, taken to the ideal limit, is Newton's First Law for rotation: **in the absence of torque, a body will spin at constant angular velocity indefinitely.**

It's worth mentioning that a vertical grindstone **must have the axle through the center of gravity.** If this is *not* the case, gravity will exert a torque, and although the wheel might keep spinning, it won't be at constant angular velocity.

In fact, the torque on such an off-center wheel is given by $\tau = Mgr_{\rm CM}\sin\theta = Mgx_{\rm CM}$, where $r_{\rm CM}$ is the distance of the center of mass from the axle, and ϑ the angle between $\vec{r}_{\rm CM}$ and the vertical.

To prove this, imagine the wheel to be made up of a large number of glued-together small masses m_i at positions $\vec{r}_i$.



As discussed in the preceding section, the torque about the axle of the weight of the small mass m_i is equal to $m_i g$ multiplied by the perpendicular distance from the axle to the line of action of the force, that is, $m_i g x_i$, taking the axle as the origin of coordinates.

Evidently the magnitude of the total gravitational torque from the imbalance

$$\tau = \sum_i m_i g x_i = Mgx_{\mathrm{CM}}$$

from the definition of the center of mass.

Kinds of Equilibrium

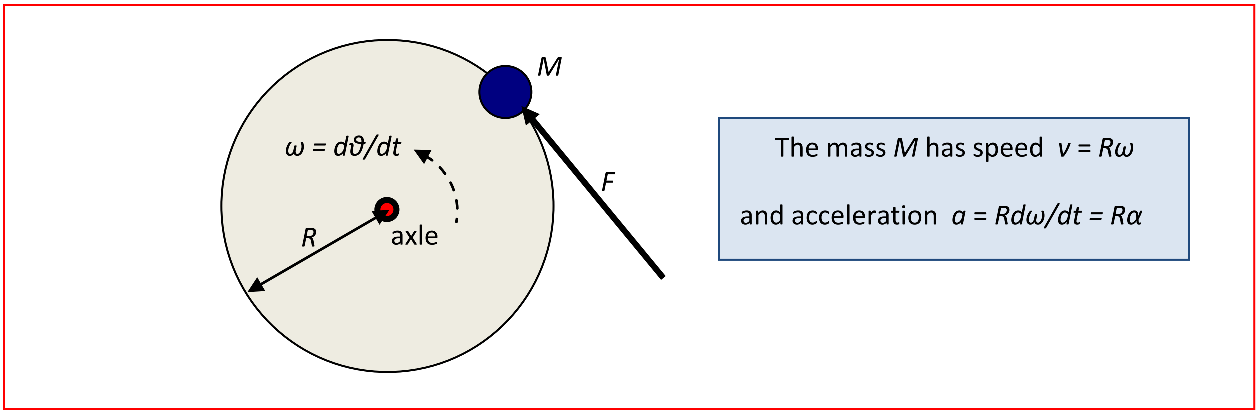
If there is no friction at the axle, the wheel can only be at rest if the center of mass lies on the vertical line through the axle. If the center of mass is directly *above* the axle, the wheel might be at rest, but the slightest push will start it turning: this is called "unstable equilibrium". If the center of mass is below the axle, a slight push to one side will cause the wheel to swing back then oscillate, a topic we'll return to later—this is "stable equilibrium". If the center of mass is exactly *at* the axle, the wheel will rotate at a steady rate—this is "neutral equilibrium". These all have analogies in linear motion: assuming no friction, unstable equilibrium is a ball perched on a mountaintop, stable is a ball sitting at the bottom of a valley, and neutral is a ball on a flat plane. In two dimensions, there are more complicated possibilities: a ball in the middle of a saddle might be stable for displacements parallel to the horse, but unstable for displacement the other way. Situations analogous to this horsy example do in fact occur in physical systems.

Rotational Equivalent of Newton's Second Law

A Simple Example: Introducing the Moment of Inertia

Newton's Second Law, in the form $\vec{F} = m\vec{a}$, states that an object undergoes linear acceleration in response to an net external force, and the mass determines the magnitude of this response. It's pretty clear by now that the angular version must be that an object undergoes angular acceleration in response to an net external torque—but it's less clear what plays the role of mass this time.

To find out, we'll look at an example of angular acceleration which is almost the same as linear acceleration.



Suppose a mass M is firmly attached to a horizontal disk of negligible mass on a frictionless vertical axle, so that the mass M must go around in a horizontal circle. Assume the mass is initially at rest, then the force F is applied. What happens in the first few moments?

Well, of course, the mass accelerates with acceleration $a = F/M$. As soon as it reaches a significant speed, the disk will begin to exert a force towards the center, to keep it on its circular path, but let's just concentrate on those first few moments before that effect becomes important.

Let's translate that initial acceleration into angular language. Since the angular velocity ω is related to the actual velocity by $v = R\omega$, the angular acceleration α must be related to the ordinary acceleration a along a tangential direction by $a = R\alpha$. So we can see how to connect the two accelerations.

Now we must connect the two forces, or, rather, the linear force F with the torque τ acting on the system mass + wheel.

The torque τ is just equal to the force F multiplied by its leverage arm R, that is, $\tau = FR$. Putting this together with $a = R\alpha$, we find that $F = Ma$ translates to:

$$\tau = FR = MaR = MR^2\,\alpha$$

Comparing this equation, $\tau = MR^2\,\alpha$, with $F = Ma$, we notice that the quantity MR^2 plays the role of mass, or inertia, in rotational dynamics for this simple case. It is called the **moment of inertia**, and labeled I. We show in the next section that for more complicated rigid bodies, $\tau = I\alpha$ is still correct, and I is a sum over the body of terms like MR^2.

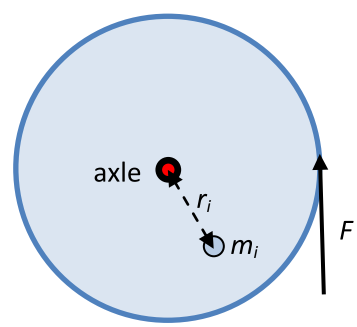
Moments of Inertia for Extended Bodies

We've only found the "moment of inertia" for a single mass attached to a light disk. What about something more complicated, for example a heavy disk?

The key to finding the moment of inertia for a solid body is to visualize it as a lot of small masses all glued together. Think specifically of a heavy wheel of radius R, rotating about a fixed axle with negligible friction. Imagine the wheel to be initially at rest, then a force F is applied to the rim tangentially, so that the torque is FR. Of course, the whole wheel begins to rotate, so think what happens to one of the small masses m_i we are imagining the wheel to be made up of. If it is at a distance r_i from the axle, and the initial angular acceleration of the wheel is α, this small mass, m_i say, must experience an initial acceleration $r_i\alpha$, so there must be a total force on it of $f_i = m_ir_i\alpha$, and therefore a torque acting on it of $f_ir_i = m_ir_i^2\alpha$.

Rigid wheel made up of many little masses m_i: when external torque FR is applied, m_i has initial linear acceleration $r_i\alpha$, so experiences torque:

$$\tau_i = f_ir_i = m_ir_i^2\alpha$$



But where, exactly does this torque come from? Ultimately, it must come from the force F being applied to a different part of the wheel, but the **immediate** cause for the small mass m_i's acceleration is just the pulls and pushes of its neighbors.

In other words, if you push on one part of the wheel, since it's a solid body its rigidity transmits the effect of the force to all the other parts, so the wheel rotates as a whole—provided of course the force is not strong enough to tear the wheel apart.

Since the torque felt by one small part of the wheel $\tau_i = m_ir_i^2\alpha$, the total torque felt by all the parts of the wheel is:

$$\tau = \Sigma\tau_i = \Sigma\, m_ir_i^2\alpha = \int dmr^2\,\alpha$$

with the integral being over all the tiny masses dm making up the wheel.

Now comes a crucial point. How do we relate this torque to the external force torque *FR*? What about all the internal forces—the fact that each part of the wheel feels forces from neighboring parts? How do we take account of this very complicated situation?

The answer is: we don't have to! By Newton's Third Law, all those internal forces are in equal opposite pairs, actions and reactions between neighboring masses. This means that when we sum over all parts of the wheel, we count all these forces, and they all cancel each other in pairs.

Therefore the total torque is just that from the external forces.

It follows that the analog of Newton's Second Law for rotation about an axle is

$$\text{Torque } \tau = I\alpha$$

where the **moment of inertia I is given by $\Sigma m_i r_i^2 = \int dmr^2$. (And remember that as always the acceleration must be measures in radians/sec^2.)**

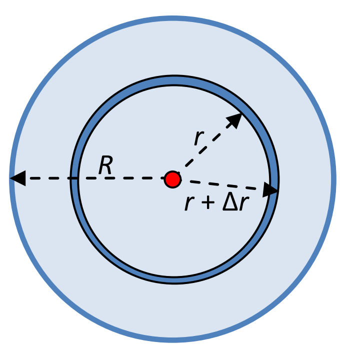
Moments of Inertia: Hoops, Disks, Cylinders, Rods.

Finding the moment of inertia I is now just a matter of doing elementary integrals. For the case of a **hoop** of mass M, where all the mass can be taken at the same distance R from the axis of rotation, no integral at all is necessary:

For a hoop, $I = MR^2$.

A **disk** can be thought of as constructed of successive hoops, like a two-dimensional onion. As a practical matter, the best approach is to use the two-dimensional density, call it ρ, (units: kg/m^2) so that the total mass of the disk is density x total area, $M = \pi R^2\rho$. Then the "hoop" which is that part of the disk between r and $r + \Delta r$ from the center has mass $m = 2\pi r\Delta r\rho$, and that hoop has moment of inertia $mr^2 = (2\pi r\Delta r\rho)r^2$.

A disk can be seen as made of concentric hoops glued together.



Then just do an integral to add up the moments of inertia of all the nested hoops from 0 to R to find the moment of inertia of the disk:

$$I_{\text{disk}} = \int_0^R 2\pi\rho r^3 dr = \tfrac{1}{2}\pi\rho R^4 = \tfrac{1}{2}MR^2.$$

The same expression works for a cylinder, which for these purposes is just a stack of disks.

What about a rod about one end? Assume its (constant) linear density is λ kg/m., so a length dx has mass λdx. Then the moment of inertia is

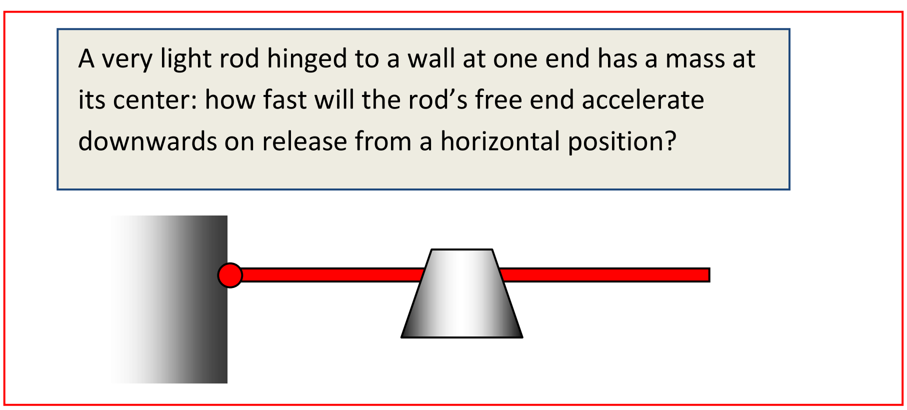
$$I = \int_0^L x^2 \lambda dx = \lambda L^3/3 = ML^2/3.$$

Suppose such a rod is hinged at one end, horizontal, and falls. What is the initial acceleration of its end?

$$\begin{aligned} \tau &= MgL/2 = I\alpha = (ML^2/3)\alpha \\ \alpha &= 3gL/2. \end{aligned}$$

So the end accelerates at $L\alpha = 3g/2$.

If it is at first sight surprising that the end of the rod accelerates downwards faster then g, consider the following system: a very light rod, hinged at one end, with a large weight attached to its middle.

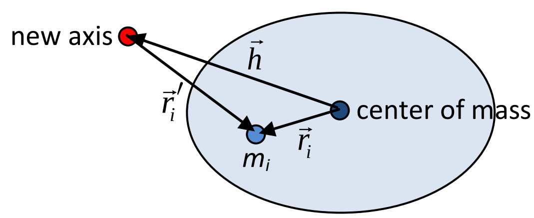


If let go from a horizontal position, the weight will completely dominate the situation, and accelerated downwards very close to g. This means the far end of the light rod is forced downwards at close to $2g$. Our example above is not that extreme, but the same idea.

The Parallel Axis Theorem

If we know the moment of inertia of an object about a line through the center of mass, the moment of inertia about any parallel line is easily found.

We'll prove this for a two-dimensional object (really a thin three-dimensional object):



The moment of inertia of the two-dimensional object about any axis perpendicular to the object is simply related to that about the parallel axis through the center of mass.

With the notation in the diagram, $I_{\text{CM}} = \sum_i m_i r_i^2$, and recall $\sum_i m_i \vec{r}_i = 0$.

Now

$$\begin{aligned} I_{\text{newaxis}} &= \sum_i m_i r_i'^2 = \sum_i m_i (\vec{r}_i - \vec{h})^2 \\ &= I_{\text{CM}} + 2\vec{h}\cdot\sum_i m_i \vec{r}_i + \sum_i m_i h^2 \\ &= I_{\text{CM}} + Mh^2. \end{aligned}$$

This is the **Parallel Axis Theorem**:

$$\boxed{I_{\text{newaxis}} = I_{\text{CM}} + Mh^2.}$$

where h is the perpendicular distance between the new axis and the parallel axis through the center of gravity. It's easy to extend this proof to a three-dimensional object: taking the parallel axes to be in the z-direction, replace $\vec{r}_i$ with (x_i, y_i).

Varying Moment of Inertia

Newton did not actually write his Second Law as $F = ma = mdv/dt$, but as $F = dp/dt$, where $p = mv$ is called the momentum, and in fact this is the correct description of nature in cases where the mass of an object varies, such as a rocket expelling fuel, or a falling raindrop growing from condensation.

Similarly, the rotational Second Law $\tau = I\alpha$ could be written more generally as

$$\tau = \frac{d}{dt}(I\omega)$$

to allow for the possibility of a varying moment of inertia, and this is much more common than a varying mass. Think of a spinning dancer or skater throwing her arms out and bringing them in, or of a diver curling up and uncurling his rotating body. The external torques are pretty small under these conditions, the variation in spin speed is largely due to the changing moment of inertia and the constancy of the total angular momentum $I\omega$. The observed variations in angular velocity confirm the validity of the more general law.

Rotational Kinetic Energy

A rotating object has kinetic energy even if its center of mass is at rest. Think of it as composed of many small masses m_i at distances r_i from the center of mass.

Then if the angular velocity is ω, m_i has speed $v_i = \omega r_i$, and kinetic energy ½ $m_i r_i^2 \omega^2$, and summing these,

KE of rotation = ½ $I\omega^2$.

If a net torque is acting, this energy will change:

$$\frac{d}{dt}\left(\tfrac{1}{2}I\omega^2\right) = I\omega\frac{d\omega}{dt} = I\omega\alpha = \omega\tau$$

so the rate of working, the power, of a torque equals the magnitude of the torque multiplied by the angular velocity, just as a force F moving something at speed v is working at a rate Fv.

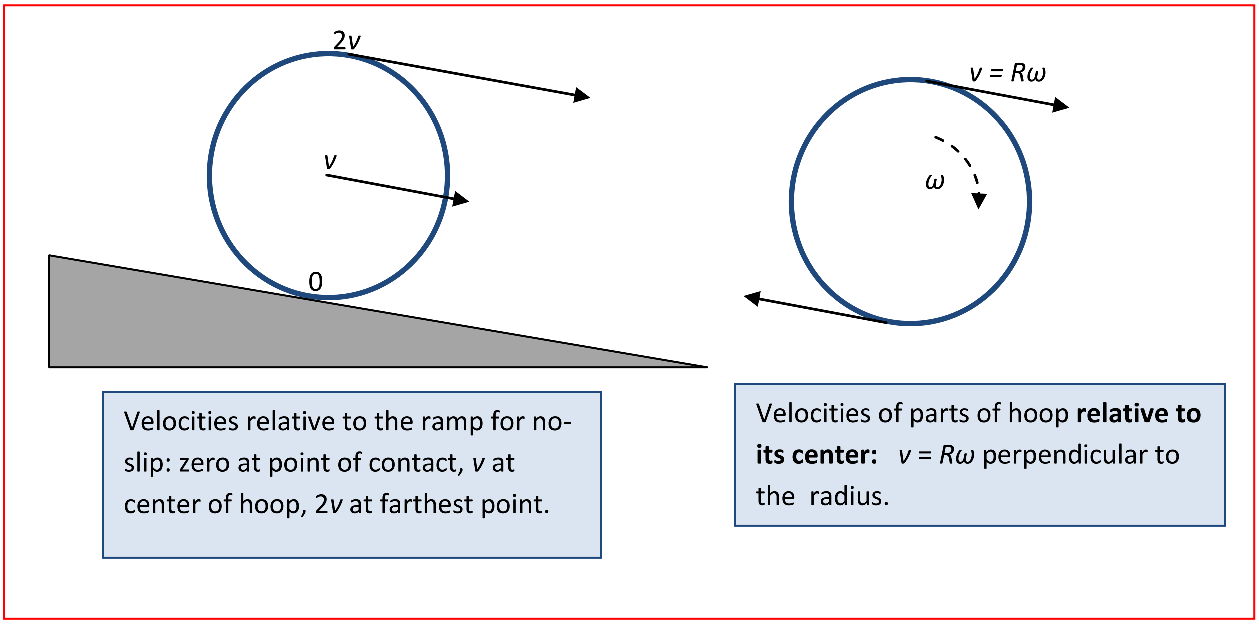
This equation can also be integrated:

$$\int d\left(\tfrac{1}{2}I\omega^2\right) = \int \omega\tau dt = \int \tau\frac{d\theta}{dt}dt = \int \tau d\theta$$

The total change in kinetic energy equals the integral of torque times angular distance, valid even if the torque is varying.

Rolling Down a Ramp

For example, we'll take the case of a hoop of radius R rolling down a ramp **without slipping**. Suppose at some instant the center of the hoop is moving with speed v. Of course, other points on the hoop are moving at different speeds. The point at the bottom in contact with the ramp isn't moving at all at that instant. But in the frame of reference in which the center of mass is momentarily stationary, that point is moving backwards at ωR. This means that the angular speed and the linear speed of the center of mass are related by $v = \omega R$.



What is the total energy of the rolling hoop? Imagine as usual that it is made up of a large number of small masses m_i. If the small mass m_i has velocity v_i, the total energy is $\frac{1}{2}\Sigma\, mv_i^2$. Since its both moving along and rolling, let's write the velocity of the small mass m_i as

$$\vec{v}_i = \vec{v}_{\mathrm{CM}} + \vec{u}_i$$

where $\vec{u}_i$ is **the velocity of the small mass m_i relative to the center of mass**.

Then the total kinetic energy of all the small masses

$$\sum_i \tfrac{1}{2} m_i \vec{v}_i^2 = \sum_i \tfrac{1}{2} m_i \left(\vec{v}_{\mathrm{CM}} + \vec{u}_i\right)^2 = \tfrac{1}{2} M v_{CM}^2 + \vec{v}_{CM} \cdot \sum_i m_i \vec{u}_i + \sum_i \tfrac{1}{2} m_i u_i^2 .$$

The first term is just the ordinary kinetic energy of linear motion, the last term is the same as the kinetic energy of rotation for the center of mass at rest. The middle term is zero, because the sum is just the linear momentum in the center of mass frame, which is zero. ($\sum_i m_i \vec{u}_i = \dfrac{d}{dt} \sum_i m_i \vec{r}_i = 0$ in the frame in which the center of mass is at rest.)

Therefore, the total energy is just the sum of the translational kinetic energy and the rotational kinetic energy:

$$E = \tfrac{1}{2} M v_{CM}^2 + \tfrac{1}{2} I \omega^2$$

where I is the moment of inertia about the center of mass.

So what does this mean for the hoop? If it's rolling at speed v, it's rotating at angular speed $\omega = v/R$, and since its moment of inertia is MR^2, its total energy is

$$E = \tfrac{1}{2} M v^2 + \tfrac{1}{2} I \omega^2 = \tfrac{1}{2} M v^2 + \tfrac{1}{2} M R^2 (v/R)^2 = M v^2.$$

So it has twice the kinetic energy of a block of the same mass sliding down at the same speed. Put another way, a block sliding down through a height h will have speed given by $v^2 = 2gh$, the hoop will only reach speed $v^2 = gh$, because it will have lost the same amount of potential energy, so must have the same **total** kinetic energy—including the rotational kinetic energy, which the block didn't have.

The equation $F = ma$ still describes its linear motion down the ramp, and after descending a distance x its velocity is given by $v^2 = 2ax$. Evidently, the rolling hoop has only **half** the acceleration of the sliding block, so must only be subject to half the force pulling it downhill. The difference is the retarding static frictional force at the point of contact: that must be just half of the gravitational force component pointing down the hill. If there isn't enough friction, the hoop will slide without increasing its angular velocity. (*Note*: the hoop has only half the acceleration of the block, but reaches a final velocity greater than half that of the block, because it accelerates for a longer time.)