Simple Harmonic Motion

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Mass on a Vertical Spring

Suppose a mass *m* is hung from a vertical spring of spring constant *k* and unstretched length *L* and is at rest with the spring stretched by x_0, that is, $mg = kx_0$. We'll use *x* here to denote the mass's displacement in the downward direction.

Now let's tug the mass downward by a further distance *x*, so *x* measures its displacement from the rest position, this generates an upward restoring force $-kx$ acting on the mass, from Hooke's Law . (The spring actually exerts a total upward force of $-k(x + x_0)$, but gravity cancels the $-kx_0$.)

So $F = ma$ for vertical motion of the mass with gravity and the spring has the form:

$$m\frac{d^2x}{dt^2} = -kx$$

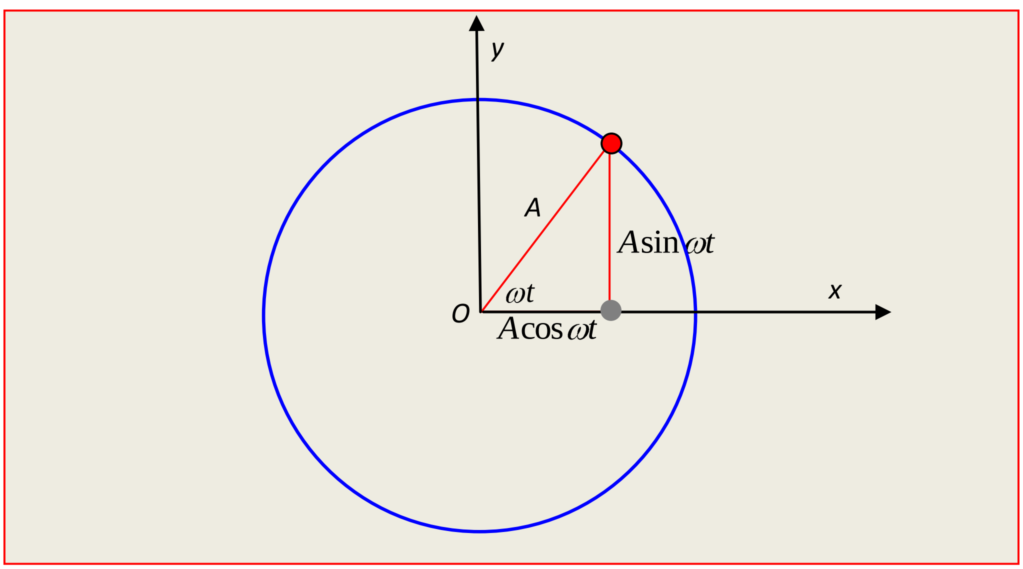
This is the differential equation that describes the motion of the mass bobbing up and down, ignoring air resistance, friction, and possible nonlinear spring behavior: this idealized motion is termed **simple harmonic motion**.

How a Shadow of Uniform Circular Motion Describes Simple Harmonic Motion

Suppose a mass *m* is moving at constant angular velocity ω around a circle of radius *A*, centered at the origin, so

$$\theta = \omega t.$$

Let's follow the shadow of this circling mass on the *x*-axis, that is, $x = A\cos\omega t$.



What is the acceleration of this shadow? The acceleration of the circling object is the vector pointing towards the center of the circle having magnitude $v^2/A = A\omega^2$. This acceleration can be split into components in the x and y directions, and the acceleration of the shadow on the x-axis is just the x-component of the total acceleration, which is just

$$\frac{d^2x}{dt^2} = -A\omega^2 \cos \omega t.$$

Putting these last two equations together, we see that the motion of the shadow $x = A\cos \omega t$ satisfies the differential equation

$$\frac{d^2x}{dt^2} = -\omega^2 x$$

which is identical to $F = ma$ for the mass bobbing on the spring, provided $\omega^2 = k/m$.

That is to say, the solution to $m\frac{d^2x}{dt^2} = -kx$ is just

$$x = A\cos\ \ \omega t + \varphi\ \ , \quad \omega = \sqrt{k/m}$$

where we've add a constant angle φ so the mass can be somewhere other than at A at $t = 0$.

Actually, we could have just written this down from a knowledge of the solutions of this differential equation, but seeing it as a shadow of steady circular motion perhaps makes it clearer.

The **period** T of the simple harmonic motion is the time for one complete oscillation, once around the circle for the circling motion, or 2π radians. This takes time

$$T = \frac{2\pi}{\omega} = 2\pi\sqrt{\frac{m}{k}}.$$

The **frequency** f is the number of complete oscillations per second, so

$$f = 1/T = \omega = 2\pi.$$

The **velocity** at any time

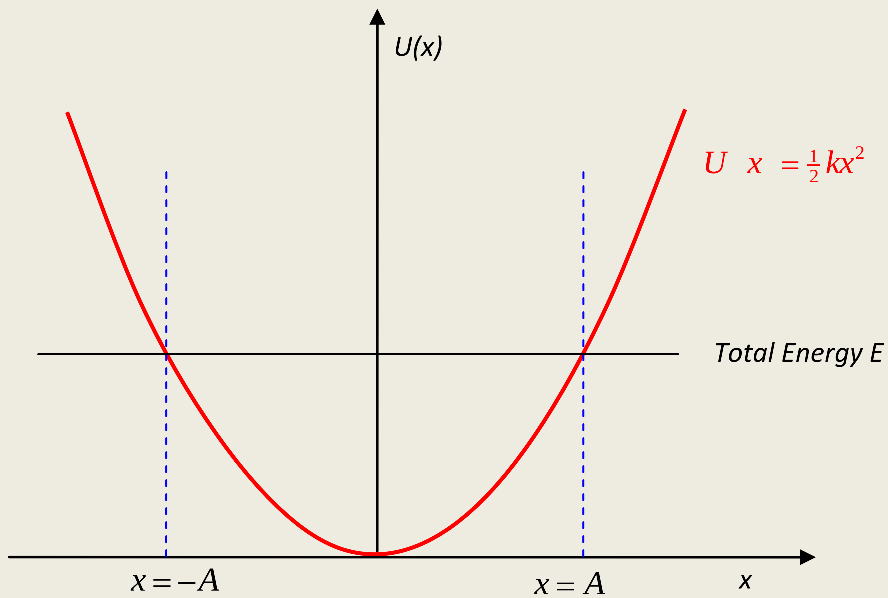
$$v = \frac{dx}{dt} = -\omega A\sin\ \ \omega t + \varphi\ \ .$$

(Check this by finding the velocity of the shadow in the steady circular motion.)

Energy in the Simple Harmonic Oscillator

For the mass oscillating on the spring, the mass has kinetic energy

$$K.E. = \tfrac{1}{2} m v^2 = \tfrac{1}{2} m \omega^2 A^2 \sin^2 (\omega t + \varphi) .$$



Potential Energy *U*(*x*) for a Simple Harmonic Oscillator.

For ***total*** energy *E*, the oscillator swings back and forth between *x* = – *A* and *x* = +*A*.

The spring exerts a force –*kx* when it is displaced from equilibrium by *x*, so the work done in stretching it by *x* is the average force times the distance,

$$P.E. = \tfrac{1}{2} k x^2 = \tfrac{1}{2} k A^2 \cos^2 (\omega t + \varphi) .$$

The kinetic and potential energies are more alike than they appear, because $k = m\omega^2$, and the total energy

$$E = K.E. + P.E. = \tfrac{1}{2} k A^2$$

using $\sin^2 + \cos^2 = 1$.

The **total energy stays constant** because there is no friction—and, must equal the potential energy stored in the spring at maximum stretch, because at that point the mass is at rest, so has zero kinetic energy.