Physics 2415 Lecture 25: Waves

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Dimensions

As a preliminary to studying waves, we'll discuss a useful trick for finding how a simple system's behavior depends on some parameter: dimensional analysis.

In mechanics, the basic units are units of length, time and mass. We denote these by L, T, and M. (It's unfortunate that we measure length in meters, m, so there is some potential for confusion there, but this is the standard notation.)

We denote the dimensionality of a quantity by putting it in square brackets. For example, velocity is distance/time, so write its dimension as $[v] = LT^{-1}$, and acceleration $[a] = LT^{-2}$. From F = ma, $[F] = MLT^{-2}$: both sides of an equation must match, dimensionally, for the equation to make sense. In fact, checking the dimensions of all terms in an equation can be very effective in finding if some term has been omitted,

Dimensional analysis can yield important physical information about a system without going through a full analysis. For example: experimentally, the time of swing of a simple pendulum varies with its length, but not with its mass. It obviously also depends on *g* (if there's no gravity, it won't swing). Without further experiment, can we find just how the time of a swing depends on length?

The answer is yes—by a simple dimensional argument. We know the time of one swing, which obviously has dimension T, can depend only on the length ℓ, of dimension L, and *g*, an acceleration, so having dimension LT^{-2}. To find a combination that is purely time, I need to divide out the L dependence.

We see that $[\ell/g] = T^2$, so $\sqrt{\ell/g}$ has dimension T: this must be the relevant factor in the equation for the period. In fact, the period is $2\pi\sqrt{\ell/g}$: dimensional analysis cannot tell us the constant multiplier, since it's dimensionless, but it does give us important physical information, such as how the period will change if we double the pendulum's length. And, we found that with very little work.

We'll be using dimensional analysis below to find dependence of wave velocities on medium parameters.

Types of Waves: Transverse and Longitudinal

Transverse: Waves on a String

The basic transverse wave is a wave on a string, including a pulse-type wave generated by flicking one end of a taut string up and down quickly. Experimentally, it's found that the resulting pulse travels down the string retaining its original shape, apart from frictional losses.

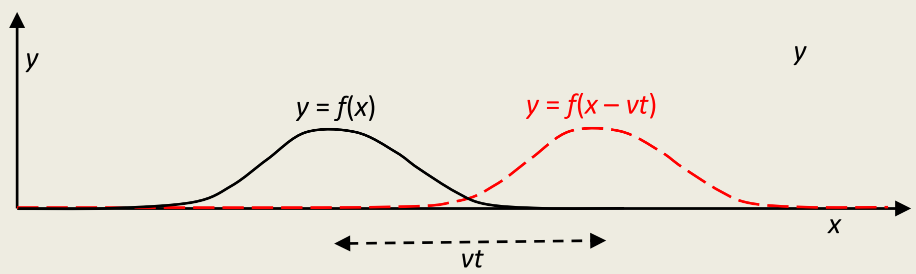
The way to describe a wave on a string mathematically is to give the transverse displacement as a function of position along the string. So, let's assume that the string at rest is stretched along the x-axis (we're ignoring gravity for now), and when a pulse is generated, at $t = 0$ the string's position is given in the (x,y) plane by a formula

$$y = f(x), \quad t = 0.$$

At a subsequent time t, the pulse has moved a distance vt to the right, but has the same shape, so is described by the same function f, but now with the function's origin shifted by vt.

So the wave form as a function of x and t is:

$$y = f(x - vt).$$



A transverse pulse traveling down a string: the form at $t = 0$ and at later time t. The function f, the shape, is unaltered.

For a pulse moving along the string to the left, the functional form would be $y = g(x - vt)$.

If two such pulses meet, it is found that provided all displacements are fairly small, the displacements simply add—the string has time-varying shape

$$y(x,t) = f(x - vt) + g(x + vt).$$

You can explore the sometimes surprising way pulses interact using the spreadsheet here.

Longitudinal: Sound Waves in a Tube

In a longitudinal wave, the particles of the medium (or small volumes of gases, etc.) oscillate back and forth as the wave passes through, but, unlike the string, in a sound wave in air, for example, the oscillating motion is along the direction of propagation of the wave.

This is well illustrated by this animation of the waves. Toggling the red lines reveals that any part of the air in the tube simply oscillates back and forth as the wave passes through, even though the wave is clearly moving to the right.

Wave Velocity: What Dimensional Analysis Tells Us

The velocity of a wave can be derived from the wave equation (see later). We won't reproduce the derivation here, we'll see what dimensional analysis can do.

First, consider a wave on a string. The relevant parameters are the tension force, called F_T by Giancoli, and the density per unit length, μ kg/meter.

Now $[F_T] = MLT^{-2}$ (it's a force) and $[\mu] = ML^{-1}$. We need to find a combination of these two that has dimensions of velocity, LT^{-1}. This means we must get rid of the M: the obvious way is to take F_T/μ, with dimension L^2T^{-2}. That's a velocity squared, so v is proportional to $\sqrt{F_T/\mu}$. In fact, it turns out to be equal to that.

For longitudinal waves in a gas, or a fluid, the relevant parameters are pressure, which is force per unit area, $[P] = MLT^{-2}L^{-2} = ML^{-1}T^{-2}$, and density, now per unit volume, $[\rho] = ML^{-3}$. Dividing one by the other gives v proportional to $\sqrt{P/\rho}$. This time the dimensional analysis doesn't give the exact result, which is $v = \sqrt{dP/d\rho} = \sqrt{B/\rho}$ where B is the bulk modulus, $B = \rho dP/d\rho$. The dimensional analysis gets the constant wrong, but gets the dependence on pressure and density right.

Notice that for an ideal gas, where $PV = nRT$, writing the density ρ as n/V moles/m^3, $P/\rho = RT$, so the speed of sound is directly proportional to $\sqrt{T}$, but doesn't depend on the density of the gas! The point is that the molecules themselves have speed proportional to $\sqrt{T}$, and they're transmitting the sound wave.

A Harmonic Wave: Energy Density and Power

Harmonic Wave Energy Density

A harmonic wave is a pure musical note, produced by an oscillating string or reed—one with no harmonics.

It's a sine wave, so at $t = 0$, say, it's

$$f(x) = A\sin\frac{2\pi x}{\lambda}$$

This is standard notation: λ is wavelength: as x increases by λ, the sine goes through a complete cycle, one wave of the wave. More commonly, the function is written in the neater form $A\sin kx$, *k is called the wave number.*

The *traveling wave*, taking it to be moving to the right, is then $A\sin k(x - vt)$, usually written

$$f(x,t) = A\sin(kx - \omega t)$$

where ω is the angular frequency.

This expression is the displacement from the straight line resting position, so *the velocity of a bit of string at x, t is in this transverse direction,* and equals $v_y(x,t) = \frac{\partial f(x,t)}{\partial t} = A\omega\cos(kx - \omega t)$.

Therefore, the kinetic energy per meter of string from this transverse motion is

$$\text{KE / meter} = \tfrac{1}{2}\rho\overline{v^2} = \tfrac{1}{2}\rho\overline{A^2\omega^2\cos^2(kx - \omega t)} = \tfrac{1}{4}\rho A^2\omega^2.$$

Now, each particle of string is oscillating as in a simple harmonic oscillator, so we argue that, like the oscillator, on average its KE is ½ its total energy. This gives

the total energy per meter of string = $\tfrac{1}{2}\rho A^2\omega^2$.

(A more convincing proof can be given by computing the potential energy as well as the kinetic energy: the potential energy is there because the string is longer as the wave passes through, and it took work against the tension to stretch it.)

Power in a Harmonic Wave

For a wave moving to the right at speed *v*, the *power* is *v* times the energy per unit length, $P = \tfrac{1}{2}v\rho A^2\omega^2$.

For example, if the traveling wave moves at 20 meters per second, and its energy is being totally absorbed as it reaches the end of the string, then each second the energy content of 20 meters of string is absorbed at the end.