home

# Physics 2415 Lecture 24: Circuits with AC Source

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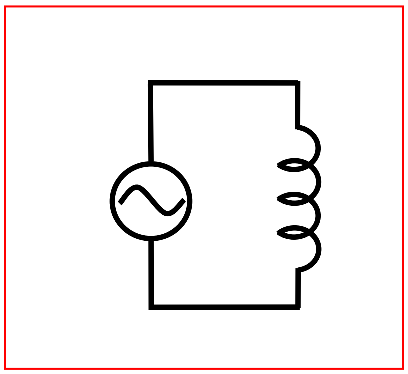
## Pure Resistance *R*

If we put an alternating voltage $V = V_0 \cos \omega t$ across a **resistor** *R*, from Ohm's law the current will be $I = I_0 \cos \omega t$ and $V_0 = I_0 R$. The power dissipation in the resistor $\overline{P} = I_0 V_0 \overline{\cos^2 \omega t} = \frac{1}{2} I_0 V_0 = V_{\text{rms}}^2 / R$.

## Pure Inductance *L*



If we put an alternating voltage across an **inductor** *L*, having zero resistance *R*, this external voltage must be exactly matched by the back emf generated.

That is, $V = L dI / dt$. So if $V = V_0 \cos \omega t$, we must have a current $I = I_0 \sin \omega t$, with $V_0 = \omega L I_0$.

This looks a lot like Ohm's law, so by analogy we write

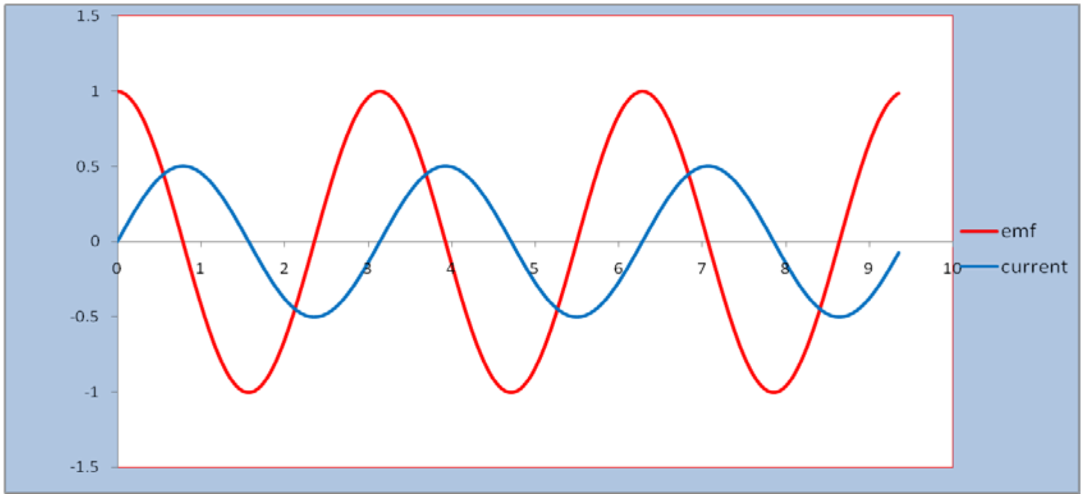
$$V_0 = I_0 X_L$$

and call $X_L$ the inductive **reactance**.

But there's a big difference from a resistance: **no power** is dissipated by a pure inductance: $\overline{P} = I_0 V_0 \overline{\sin \omega t \cos \omega t} = \frac{1}{2} I_0 V_0 \overline{\sin 2\omega t} = 0$.

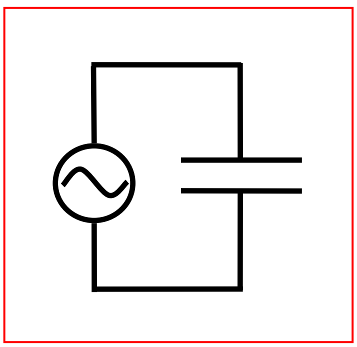
It's worth plotting voltage and current as functions of time on the same graph. For $V = V_0 \cos \omega t$, at $t = 0$ the voltage is at its maximum positive value, so the current is increasing at a maximum rate. One-quarter of a cycle later ( $\omega t = \pi / 2$ ) the external applied voltage is zero, so at that instant the current is not changing—in fact, the current has reached its maximum positive value. So we see the current reaches its maximum one quarter of a cycle (or $\pi$/2 or 90°) after the maximum voltage: we say the current lags behind the voltage by 90°.

**Note:** the graph below, and the subsequent ones on AC circuits, were generated by an Excel spreadsheet available for download from my 2415 Home Page. Try it—it's a good way to explore these circuits!




**Current and emf in a purely inductive AC circuit: emf leads current by a quarter cycle (ELI).**

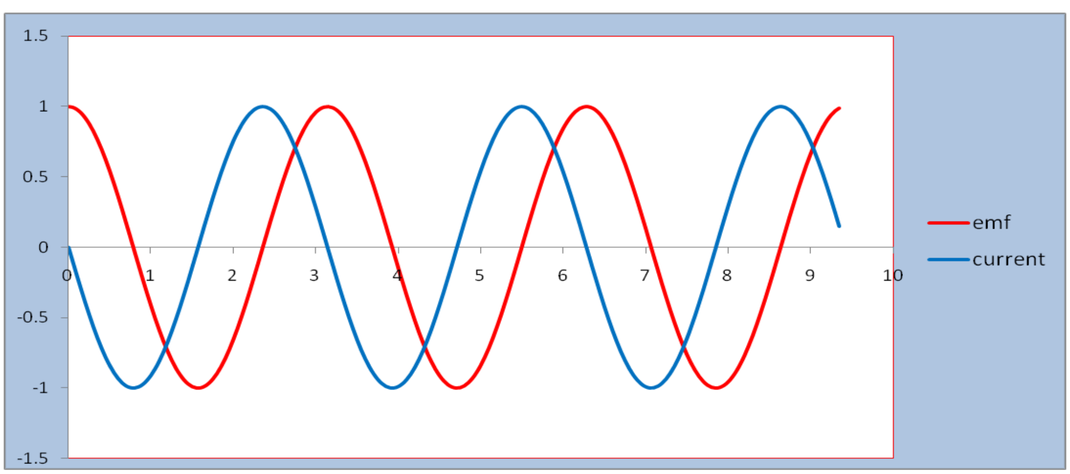
## Pure Capacitance $C$



Putting $V = V_0 \cos \omega t$ across a capacitor, again with no resistance (or inductance) in the circuit, at $t = 0$ the voltage is at its maximum positive value, which means the capacitance contains its maximum positive charge in the cycle. At that moment, the current must be zero: the capacitance has charged up fully, and is about to begin discharging as it follows the cycle.

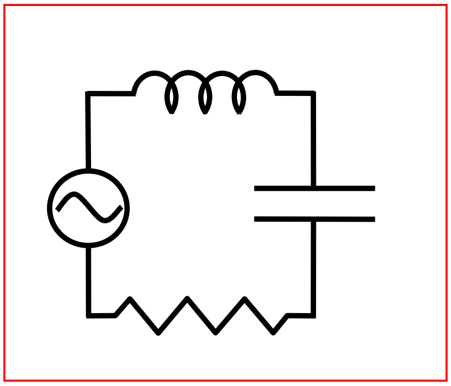
The equations are $V = V_0 \cos \omega t = Q / C$ and $I = dQ / dt = -\omega C V_0 \sin \omega t = -I_0 \sin \omega t$.

We write $V_0 = I_0 X_C$ where $X_C$ is the **capacitive reactance** $X_C = 1 / \omega C$.




**Current and emf in a purely capacitative AC circuit: current leads emf by a quarter cycle (ICE).**

## Driven LRC Circuit



The equation describing charge variation as a function of time is:

$$L\frac{d^2Q}{dt^2}+R\frac{dQ}{dt}+\frac{Q}{C}=V_0\cos\omega t$$

where we've arbitrarily fixed time zero to correspond to a maximum of the driving field. Obviously, if the AC field is turned on at time zero, there is some transient behavior before it settles down. That might be important in some situations, but we're only going to analyze the "steady state" situation, the rhythm the system settles into after the AC has been on for many cycles.

The equation can be solved (most simply using complex numbers); we'll skip the derivation and present the solution for the current. As one would expect, it has the same period of variation as the driving emf, but is not in general in phase. Explore the solutions using the spreadsheet!

The current is $I=I_0\cos\left(\omega t-\varphi\right)$ where

$$I_0=\frac{V_0}{\sqrt{\left(X_L-X_C\right)^2+R^2}}=\frac{V_0}{\sqrt{\left(\omega L-\left(1/\omega C\right)\right)^2+R^2}},\quad \varphi=\tan^{-1}\left(\frac{X_L-X_C}{R}\right).$$

(You can of course check that this is a solution by plugging it into the equation.)

The form of $\varphi$ tells us immediately that for a purely inductive circuit, the emf leads the current (ELI), for a purely capacitative circuit, current leads emf (ICE).

## Resonance

Something surprising happens if $\omega L=1/\omega C$—there is no phase lag, and the current amplitude is given by Ohm's law for the resistor alone! Notice this is the same value of $\omega$ at which a pure *LC* circuit oscillates. This is resonance: the driving frequency coincides with the natural frequency of the circuit, and the amplitude of the oscillations is limited only by the damping—the resistance.

Another way to understand this is to remember that the current through all three elements (*L*, *C*, *R*) of the circuit has the same phase. From Ohm's law, the voltage drop across the resistor oscillates exactly in phase with the current. The voltage change across the inductance has phase 90° *ahead* of the current's phase, that across the capacitor has phase 90° *behind* the current. This means that the voltage changes across the inductance and the capacitor are 180° out of phase—meaning they are opposite, so if the amplitudes are equal, they'll exactly cancel! (Check this on the spreadsheet.)

## Power Factor in an AC Circuit

As usual, $\overline{P}=\overline{VI}$ —in this case, using a standard trig identity, and $\overline{\cos^2\omega t}=\frac{1}{2}$, $\overline{\cos\omega t\sin\omega t}=0$, we have

$$\begin{aligned}\overline{P}&=\overline{VI}=V_0I_0\overline{\cos\omega t\cos\left(\omega t-\varphi\right)}\\&=V_0I_0\overline{\cos\omega t\left(\cos\omega t\cos\varphi+\sin\omega t\sin\varphi\right)}\\&=\tfrac{1}{2}V_0I_0\cos\varphi=V_{\text{rms}}I_{\text{rms}}\cos\varphi.\end{aligned}$$