18. Driven Oscillator

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Consider a one-dimensional simple harmonic oscillator with a variable external force acting, so the equation of motion is

$$\ddot{x} + \omega^2 x = F(t)/m,$$

which would come from the Lagrangian

$$L = \tfrac{1}{2}m\dot{x}^2 - \tfrac{1}{2}kx^2 + xF(t).$$

(Landau "derives" this as the leading order non-constant term in a time-dependent external potential.)

The general solution of the differential equation is $x = x_0 + x_1$, where $x_0 = a\cos(\omega t + \alpha)$, the solution of the homogeneous equation, and x_1 is some particular integral of the inhomogeneous equation.

An important case is that of a periodic driving force $F(t) = f\cos(\gamma t + \beta)$. A trial solution $x_1(t) = b\cos(\gamma t + \beta)$ yields $b = f/m(\omega^2 - \gamma^2)$ so

$$x(t) = a\cos(\omega t + \alpha) + \frac{f}{m(\omega^2 - \gamma^2)}\cos(\gamma t + \beta).$$

But what happens when $\gamma = \omega$? To find out, take part of the first solution into the second, that is,

$$x(t) = a'\cos(\omega t + \alpha') + \frac{f}{m(\omega^2 - \gamma^2)}\left[\cos(\gamma t + \beta) - \cos(\omega t + \beta)\right].$$

The second term now goes to $0/0$ as $\gamma \to \omega$, so becomes the ratio of its first derivatives with respect to ω (or, equivalently, γ).

$$x(t) = a'\cos(\omega t + \alpha') + \frac{f}{2m\omega}t\sin(\omega t + \beta).$$

The amplitude of the oscillations grows linearly with time. Obviously, this small oscillations theory will crash eventually.

But what if the external force frequency is slightly *off* resonance?

Then (real part understood)

$$x = Ae^{i\omega t} + Be^{i(\omega+\varepsilon)t} = \left(A + Be^{i\varepsilon t}\right)e^{i\omega t}, \quad A = ae^{i\alpha}, \quad B = be^{i\beta}$$

with a, b, α, β real.

The wave amplitude squared

$$C^2 = \left|A + Be^{i\varepsilon t}\right|^2 = a^2 + b^2 + 2ab\cos\left(\varepsilon t + \beta - \alpha\right).$$

We're seeing beats, with beat frequency ε. Note that if the oscillator begins at the origin, $x\left(t=0\right)=0$, then $A+B=0$ and the amplitude periodically goes to zero, this evidently only occurs when $|A|=|B|$.

Energy is exchanged back and forth with the driving external force.

More General Energy Exchange

We'll derive a formula for the energy fed into an oscillator by an *arbitrary* time-dependent external force.

The equation of motion can be written

$$\frac{d}{dt}(\dot{x} + i\omega x) - i\omega(\dot{x} + i\omega x) = \frac{1}{m}F(t)$$

and defining $\xi = \dot{x} + i\omega x$, this is

$$d\xi / dt - i\omega\xi = F(t)/m.$$

This first-order equation integrates to

$$\xi(t) = e^{i\omega t}\left(\int_0^t \frac{1}{m}F(t')e^{-i\omega t'}dt' + \xi_0\right)$$

The energy of the oscillator is

$$E = \tfrac{1}{2}m\left(\dot{x}^2 + \omega^2 x^2\right) = \tfrac{1}{2}m|\xi|^2$$

So if we drive the oscillator over all time, with beginning energy zero,

$$E = \frac{1}{2m}\left|\int_{-\infty}^{\infty} F(t)e^{-i\omega t}dt\right|^2.$$

This is equivalent to the quantum mechanical time-dependent perturbation theory result: ξ, ξ^* are equivalent to the annihilation and creation operators.

Damped Driven Oscillator

The linear damped driven oscillator:

$$\ddot{x} + 2\lambda\dot{x} + \omega_0^2 x = (f/m)e^{i\Omega t}.$$

(Following Landau's notation here—note it means the actual frictional drag force is $2\lambda m\dot{x}$)

Looking near resonance for steady state solutions at the driving frequency, with amplitude b, phase lag δ, that is, $x(t) = be^{i(\Omega t+\delta)}$, we find

$$be^{i\delta}\left(-\Omega^2 + 2i\lambda\Omega + \omega_0^2\right) = \left(f/m\right).$$

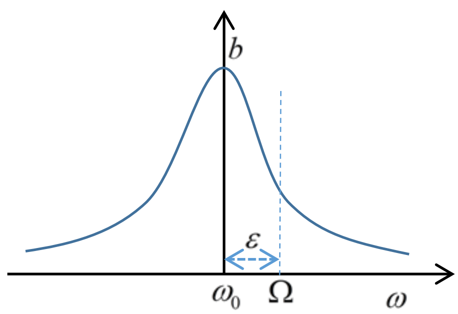
For a near-resonant driving frequency $\Omega = \omega_0 + \varepsilon$, and assuming the damping to be sufficiently small that we can drop the $\varepsilon\lambda$ term along with ε^2, the leading order terms give

$$be^{i\delta} = -f/2m\left(\varepsilon - i\lambda\right)\omega_0,$$

so the response, the dependence of amplitude of oscillation on frequency, is to this accuracy

$$b = \frac{f}{2m\omega_0\sqrt{\left(\Omega-\omega_0\right)^2+\lambda^2}} = \frac{f}{2m\omega_0\sqrt{\varepsilon^2+\lambda^2}}.$$

(We might also note that the resonant frequency is itself lowered by the damping, but this is another second-order effect we ignore here.)



The rate of absorption of energy equals the frictional loss. The friction force $2\lambda m\dot{x}$ on the mass moving at $\dot{x}$ is doing work at a rate:

$$2\lambda m\overline{\dot{x}^2} = \lambda m b^2\Omega^2.$$

The half width of the resonance curve as a function of driving frequency Ω is given by the damping. The total area under the curve is independent of damping.

For future use, we'll write the above equation for the amplitude as $b^2\left(\varepsilon^2+\lambda^2\right) = \dfrac{f^2}{4m^2\omega_0^2}$.

The Excel spreadsheet for a driven damped oscillator I showed in class can be downloaded here.