# 24. Moments of Inertia: Examples

Michael Fowler

Molecules

The moment of inertia of the hydrogen molecule was historically important. It's trivial to find: the nuclei (protons) have 99.95% of the mass, so a classical picture of two point masses m a fixed distance a apart gives $I = \frac{1}{2}ma^2$. In the nineteenth century, the mystery was that equipartition of energy, which gave an excellent account of the specific heats of almost all gases, didn't work for hydrogen—at low temperatures, apparently these diatomic molecules didn't spin around, even though they constantly collided with each other. The resolution was that the moment of inertia was so low that a lot of energy was needed to excite the first quantized angular momentum state, $L = \hbar$. This was not the case for heavier diatomic gases, since the energy of the lowest angular momentum state $E = L^2 / 2I = \hbar^2 / 2I$, is lower for molecules with bigger moments of inertia .

Here's a simple planar molecule:
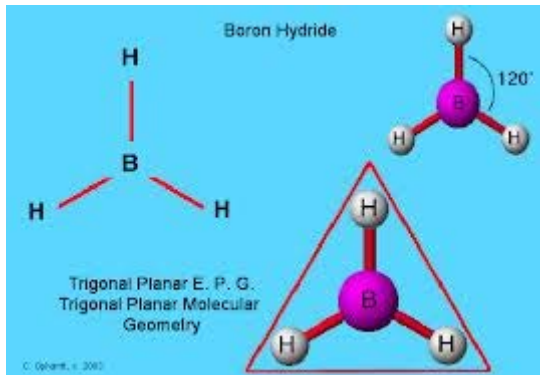
Obviously, one principal axis is through the centroid, perpendicular to the plane. We've also established that any axis of symmetry is a principal axis, so there are evidently three principal axes in the plane, one along each bond! The only interpretation is that there is a degeneracy: there are two equal-value principal axes in the plane, and any two perpendicular axes will be fine. The moment of inertial about either of these axes will be one-half that about the perpendicular-to-the-plane axis.
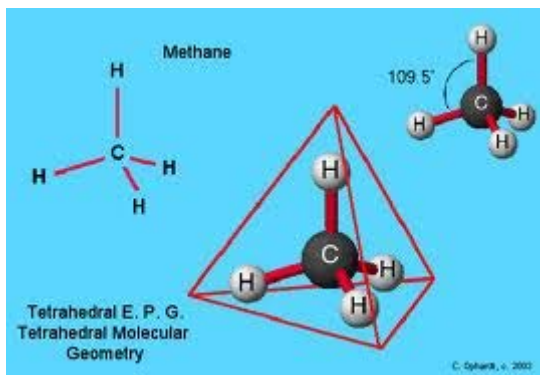
What about a symmetrical three dimensional molecule?

Here we have four obvious principal axes: only possible if we have spherical degeneracy, meaning all three principal axes have the same moment of inertia.

Various Shapes

A thin rod, linear mass density λ, length ℓ: $I = 2\int_0^{\ell/2} \lambda x^2 dx = 2\lambda \ell^3 / 24 = \frac{1}{12}m\ell^2$.

A square of mass m, side ℓ, about an axis in its plane, through the center, perpendicular to a side: $\frac{1}{12}m\ell^2$. (It's just a row of rods.) in fact, the moment is the same about any line in the plane through the center, from the symmetry, and the moment about a line perpendicular to the plane through the center is twice this—that formula will then give the moment of inertia of a cube, about any axis through its center.

A disc of mass M, radius a and surface density σ has $I = \int_0^a r^2 \cdot \sigma \cdot 2\pi r dr = \frac{1}{2}\pi a^4 \sigma = \frac{1}{2}Ma^2$. This is also correct for a cylinder (think of it as a stack of discs) about its axis.

A disc about a line through its center in its plane must be $\frac{1}{4}Ma^2$ from the perpendicular axis theorem. A solid cylinder about a line through its center perpendicular to its main axis can be regarded as a stack of discs, of radius a, height h, taking the mass of a disc as ρdz, and using the parallel axes theorem,

$$I = 2\int_0^{h/2} \rho dz \left(\tfrac{1}{4}a^2 + z^2\right) = \tfrac{1}{4}Ma^2 + \tfrac{1}{12}Mh^2.$$

For a sphere, a stack of discs of varying radii, $I = \int_{-a}^a dz \tfrac{1}{2}\rho\pi \left(a^2 - z^2\right)^2 = \tfrac{8}{15}\rho\pi a^5 = \tfrac{2}{5}Ma^2$.
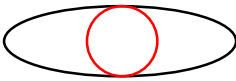
An ellipsoid of revolution and a sphere of the same mass and radius clearly have the same motion of inertial about their common axis (shown).
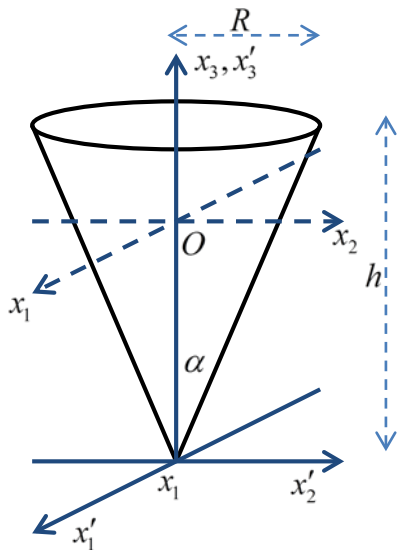
Moments of Inertia of a Cone

Following Landau, we take height h and base radius R and semivertical angle α so that $R = h\tan\alpha$.

As a preliminary, the volume of the cone is

$$V = \int_0^h \pi r^2 dz = \int_0^h \pi \left(\frac{Rz}{h}\right)^2 dz = \tfrac{1}{3}\pi R^2 h.$$

The center of mass is distance a from the vertex, where

$$aV = a \cdot \tfrac{1}{3}\pi R^2 h = \int_0^h z dV = \int_0^h \pi z \left(\frac{Rz}{h}\right)^2 dz = \tfrac{1}{4}\pi R^2 h^2, \quad a = \tfrac{3}{4}h.$$

The moment of inertia about the axis x_3 of the cone is (taking density ρ) that of a stack of discs each

having mass $m(dz) = \pi r^2 \rho \, dz = \pi \left(\dfrac{Rz}{h} \right)^2 \rho \, dz$ and moment of inertia $I(dz) = \tfrac{1}{2} m(dz) r^2$:

$$\int_0^h \tfrac{1}{2} \pi \rho \left(\frac{Rz}{h} \right)^4 dz = \tfrac{1}{10} \pi \rho R^4 h = \tfrac{3}{10} MR^2 .$$
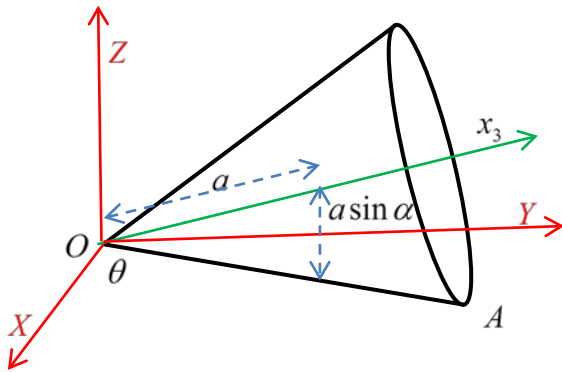
The moment of inertia about the axis x_1' through the vertex, perpendicular to the central axis, can be

calculated using the stack-of-discs parallel axis approach, the discs having mass $\pi \rho \left(\dfrac{Rz}{h} \right)^2 dz$, it is

$$\int_0^h \pi \rho \left(\frac{Rz}{h} \right)^2 \left[\tfrac{1}{4} \left(\frac{Rz}{h} \right)^2 + z^2 \right] dz = \tfrac{1}{20} \pi \rho R^4 h + \tfrac{1}{5} \pi \rho R^2 h^3 = \tfrac{3}{20} MR^2 + \tfrac{3}{5} Mh^2 .$$

Analyzing Rolling Motion

Kinetic Energy of a Cone Rolling on a Plane
(*This is from Landau.*)

The cone rolls without slipping on the horizontal XY plane. The momentary line of contact with the plane is OA, at an angle θ in the horizontal plane from the X axis.

The important point is that this line of contact, *regarded as part of the rolling cone*, is momentarily at rest when it's in contact with the plane. This means that, *at that moment*, the cone is rotating about the stationary line OA. Therefore, the angular velocity vector $\vec{\Omega}$ points along OA .

 Taking the cone to have semi-vertical angle α (meaning this is the angle between OA and the central axis of the cone) the center of mass, which is a distance a from the vertex, and on the central line, moves along a circle at height $a \sin \alpha$ above the plane, this circle being centered on the Z axis, and having radius $a \cos \alpha$. The center of mass moves at velocity $V = \dot{\theta} a \cos \alpha$, so contributes translational kinetic energy $\tfrac{1}{2} MV^2 = \tfrac{1}{2} M \dot{\theta}^2 a^2 \cos^2 \alpha$.

Now visualize the rolling cone turning around the momentarily fixed line OA: the center of mass, at height $a\sin\alpha$, moves at V, so the angular velocity $\Omega = \dfrac{V}{a\sin\alpha} = \dot{\theta}\cot\alpha.$

Now define a new set of axes with origin O: one, x_3, is the cone's own center line, another, x_2, is perpendicular to that *and* to OA, this determines x_1. (For these last two, since they're through the vertex, the moment of inertia is the one worked out in the previous section.)

Since $\vec{\Omega}$ is along OA, its components with respect to these axes $\left(x_1, x_2, x_3\right)$ are $\left(\Omega\sin\alpha,\ 0,\ \Omega\cos\alpha\right)$.

The total kinetic energy is

$$T = \tfrac{1}{2}M\dot{\theta}^2 a^2 \cos^2\alpha + \tfrac{1}{2}I_1\dot{\theta}^2\cos^2\alpha + \tfrac{1}{2}I_3\dot{\theta}^2\frac{\cos^4\alpha}{\sin^2\alpha}$$
$$= 3Mh^2\dot{\theta}^2\left(1 + 5\cos^2\alpha\right)/40,$$

using $I_1 = \tfrac{3}{20}MR^2 + \tfrac{3}{5}Mh^2,\ I_3 = \tfrac{3}{10}MR^2,\ a = \tfrac{3}{4}h,\ R = h\tan\alpha.$

Rolling Without Slipping: Two Views

Think of a hoop, mass M radius R, rolling along a flat plane at speed V. It has translational kinetic energy $\tfrac{1}{2}MV^2$, angular velocity $\Omega = V/R$, and moment of inertia $I = MR^2$ so its angular kinetic energy $\tfrac{1}{2}I\Omega^2 = \tfrac{1}{2}MV^2$, and its total kinetic energy is MV^2.

But we could also have thought of it as *rotating about the point of contact*—remember, that point of the hoop is momentarily at rest. The angular velocity would again be Ω, but now with moment of inertia, from the parallel axes theorem, $I = MR^2 + MR^2 = 2MR^2$, giving same total kinetic energy, but now all rotational.

Cylinder Rolling Inside another Cylinder

Now consider a solid cylinder radius a rolling inside a hollow cylinder radius R, angular distance from the lowest point θ, the solid cylinder axis moving at $V = (R-a)\dot{\theta}$, and therefore having angular velocity (compute about the point of contact) $\Omega = V/a$.
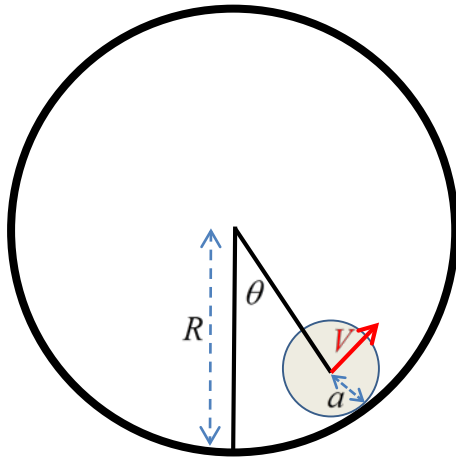
The kinetic energy is

$$\tfrac{1}{2}MV^2 + \tfrac{1}{2}I(V/a)^2 = \tfrac{1}{2}\left(M + \frac{I}{a^2}\right)(R-a)^2\dot{\theta}^2$$

The potential energy is $-Mg(R-a)\cos\theta$

The Lagrangian $L = T - V$, the equation of motion is

$$\left(M + \frac{I}{a^2}\right)(R-a)^2\ddot{\theta} = -Mg(R-a)\sin\theta \cong -Mg(R-a)\theta,$$

so small oscillations are at frequency $\omega = \sqrt{\dfrac{g}{\left(1 + \dfrac{I}{Ma^2}\right)(R-a)}}$.