Momentum, Work and Energy

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Momentum

At this point, we introduce some further concepts that will prove useful in describing motion. The first of these, *momentum*, was actually introduced by the French scientist and philosopher Descartes before Newton. Descartes' idea is best understood by considering a simple example: think first about someone (weighing say 45 kg) standing motionless on high quality (frictionless) rollerskates on a level smooth floor. A 5 kg medicine ball is thrown directly at her by someone standing in front of her, and only a short distance away, so that we can take the ball's flight to be close to horizontal. She catches and holds it, and because of its impact begins to roll backwards. Notice we've chosen her weight so that, conveniently, she plus the ball weigh just ten times what the ball weighs by itself. What is found on doing this experiment carefully is that after the catch, she plus the ball roll backwards at just one-tenth the speed the ball was moving just before she caught it, so if the ball was thrown at 5 meters per second, she will roll backwards at one-half meter per second after the catch. It is tempting to conclude that the "total amount of motion" is the same before and after her catching the ball, since we end up with ten times the mass moving at one-tenth the speed.

Considerations and experiments like this led Descartes to invent the concept of "momentum", meaning "amount of motion", and to state that for a moving body the momentum was just the product of the mass of the body and its speed. Momentum is traditionally labeled by the letter p, so his definition was:

$$\text{momentum} = p = mv$$

for a body having mass m and moving at speed v. It is then obvious that in the above scenario of the woman catching the medicine ball, total "momentum" is the same before and after the catch. Initially, only the ball had momentum, an amount 5x5 = 25 in suitable units, since its mass is 5kg and its speed is 5 meters per second. After the catch, there is a total mass of 50kg moving at a speed of 0.5 meters per second, so the final momentum is 0.5x50 = 25, the total final amount is equal to the total initial amount. We have just invented these figures, of course, but they reflect what is observed experimentally.

There is however a problem here—obviously one can imagine collisions in which the "total amount of motion", as defined above, is definitely *not* the same before and after. What about two people on rollerskates, of equal weight, coming directly towards each other at equal but opposite velocities—and when they meet they put their hands together and come to a complete halt? Clearly in this situation there was plenty of motion before the collision and none afterwards, so the "total amount of motion" definitely doesn't stay the same! In physics language, it is “not conserved”. Descartes was hung up on this problem a long time, but was rescued by a Dutchman, Christian Huygens, who pointed out that the problem could be solved in a consistent fashion if one did not insist that the “quantity of motion” be positive.

In other words, *if something moving to the right was taken to have positive momentum, then one should consider something moving to the left to have negative momentum*. With this convention, two people of equal mass coming together from opposite directions at the same speed would have total momentum *zero*, so if they came to a complete halt after meeting, as described above, the total momentum before the collision would be the same as the total after—that is, zero—and momentum *would* be conserved.

Of course, in the discussion above we are restricting ourselves to motions along a single line. It should be apparent that to get a definition of momentum that is conserved in collisions what Huygens really did was to tell Descartes he should replace speed by *velocity* in his definition of momentum. It is a natural extension of this notion to think of momentum as defined by

momentum = mass x *velocity*

in general, so, *since velocity is a vector, momentum is also a vector*, pointing in the same direction as the velocity, of course.

It turns out experimentally that in *any* collision between two objects (where no interaction with third objects, such as surfaces, interferes), the total momentum before the collision is the same as the total momentum after the collision. It doesn’t matter if the two objects stick together on colliding or bounce off, or what kind of forces they exert on each other, so conservation of momentum is a very general rule, quite independent of details of the collision.

Momentum Conservation and Newton’s Laws

As we have discussed above, Descartes introduced the concept of momentum, and the general principle of conservation of momentum in collisions, before Newton’s time. However, it turns out that conservation of momentum can be deduced from Newton’s laws. Newton’s laws in principle fully describe all collision-type phenomena, and therefore must contain momentum conservation.

To understand how this comes about, consider first Newton’s Second Law relating the acceleration a of a body of mass m with an external force F acting on it:

$F = ma$, or force = mass x acceleration

Recall that acceleration is rate of change of velocity, so we can rewrite the Second Law:

force = mass x rate of change of velocity.

Now, the momentum is *mv*, mass x velocity. This means for an object having constant mass (which is almost always the case, of course!)

rate of change of momentum = mass x rate of change of velocity.

This means that Newton's Second Law can be rewritten:

force = rate of change of momentum.

Now think of a collision, or any kind of interaction, between two objects *A* and *B*, say. From Newton's Third Law, the force *A* feels from *B* is of equal magnitude to the force *B* feels from *A*, but in the opposite direction. Since (as we have just shown) force = rate of change of momentum, it follows that throughout the interaction process the rate of change of momentum of *A* is exactly opposite to the rate of change of momentum of *B*. In other words, since these are vectors, they are of equal length but pointing in opposite directions. This means that for every bit of momentum *A* gains, *B* gains the negative of that. In other words, *B loses* momentum at exactly the rate *A gains* momentum so their *total* momentum remains the same. But this is true throughout the interaction process, from beginning to end. Therefore, the total momentum at the end must be what it was at the beginning.

You may be thinking at this point: so what? We already know that Newton's laws are obeyed throughout, so why dwell on one special consequence of them? The answer is that although we know Newton's laws are obeyed, this may not be much use to us in an actual case of two complicated objects colliding, because we may not be able to figure out what the forces are. Nevertheless, we *do* know that momentum will be conserved anyway, so if, for example, the two objects stick together, and no bits fly off, we can find their final velocity just from momentum conservation, without knowing any details of the collision.

Work

The word "work" as used in physics has a narrower meaning than it does in everyday life. First, it only refers to physical work, of course, and second, something has to be accomplished. If you lift up a box of books from the floor and put it on a shelf, you've done work, as defined in physics, if the box is too heavy and you tug at it until you're worn out but it doesn't move, that doesn't count as work.

Technically, work is done when a force pushes something and the object moves some distance in the direction it's being pushed (pulled is ok, too). Consider lifting the box of books to a high shelf. If you lift the box at a steady speed, the force you are exerting is just balancing off gravity, the weight of the box, otherwise the box would be accelerating. (Of course, initially you'd have to exert a little bit more force to get it going, and then at the end a little less, as the box comes to rest at the height of the shelf.) It's obvious that you will have to do twice as much work to raise a box of twice the weight, so the work done is proportional to the force you exert. It's also clear that the work done depends on how high the shelf is. Putting these together, the definition of work is:

work = force x distance

where only distance traveled in the direction the force is pushing counts. With this definition, carrying the box of books across the room from one shelf to another of equal height doesn't count as work, because even though your arms have to exert a force upwards to keep the box from falling to the floor, you do not move the box in the direction of that force, that is, upwards.

To get a more quantitative idea of how much work is being done, we need to have some units to measure work. Defining work as force x distance, as usual we will measure distance in meters, but we haven't so far talked about units for force. The simplest way to think of a unit of force is in terms of Newton's Second Law, force = mass x acceleration. The natural "unit force" would be that force which, pushing a unit mass (one kilogram) with no friction of other forces present, accelerates the mass at one meter per second per second, so after two seconds the mass is moving at two meters per second, etc. *This unit of force is called one* ***newton*** (as we discussed in an earlier lecture). Note that a one kilogram mass, when dropped, accelerates downwards at ten meters per second per second. This means that its weight, its gravitational attraction towards the earth, must be equal to ten newtons. From this we can figure out that a one newton force equals the weight of 100 grams, just less than a quarter of a pound, a stick of butter.

The downward acceleration of a freely falling object, ten meters per second per second, is often written g for short. (To be precise, g = 9.8 meters per second per second, and in fact varies somewhat over the earth's surface, but this adds complication without illumination, so we shall always take it to be 10.) If we have a mass of m kilograms, say, we know its weight will accelerate it at g if it's dropped, so its weight is a force of magnitude mg, from Newton's Second Law.

Now back to *work*. Since work is force x distance, the natural "unit of work" would be the work done be a force of one newton pushing a distance of one meter. In other words (approximately) lifting a stick of butter three feet. *This unit of work is called one* ***joule***, in honor of an English brewer.

Finally, it is useful to have a unit for *rate of working*, also called "power". The natural unit of "rate of working" is manifestly one joule per second, and this is called one *watt*. To get some feeling for rate of work, consider walking upstairs. A typical step is eight inches, or one-fifth of a meter, so you will gain altitude at, say, two-fifths of a meter per second. Your weight is, say (put in your own weight here!) 70 kg. (for me) multiplied by 10 to get it in newtons, so it's 700 newtons. The rate of working then is 700 x 2/5, or 280 watts. Most people can't work at that rate for very long. A common English unit of power is the *horsepower*, which is 746 watts.

Energy

Energy is the ability to do work.

For example, it takes work to drive a nail into a piece of wood—a force has to push the nail a certain distance, against the resistance of the wood. A moving hammer, hitting the nail, can drive it in. A stationary hammer placed on the nail does nothing. The moving hammer has energy—the ability to drive the nail in—because it's moving. This hammer energy is called "*kinetic energy*". Kinetic is just the Greek word for *motion*, it's the root word for cinema, meaning *movies*.

Another way to drive the nail in, if you have a good aim, might be to simply drop the hammer onto the nail from some suitable height. By the time the hammer reaches the nail, it will have kinetic energy. It has this energy, of course, because the force of gravity (its weight) accelerated it as it came down. But this energy didn't come from nowhere. Work had to be done in the first place to lift the hammer to the height from which it was dropped onto the nail. In fact, the work done in the initial lifting, force x distance, is just the weight of the hammer multiplied by the distance it is raised, in joules. But this is exactly the same amount of work as gravity does on the hammer in speeding it up during its fall onto the nail. Therefore, while the hammer is at the top, waiting to be dropped, it can be thought of as storing the work that was done in lifting it, which is ready to be released at any time. This "stored work" is called *potential energy*, since it has the *potential* of being transformed into kinetic energy just by releasing the hammer.

To give an example, suppose we have a hammer of mass 2 kg, and we lift it up through 5 meters. The hammer's weight, the force of gravity, is 20 newtons (recall it would accelerate at 10 meters per second per second under gravity, like anything else) so the work done in lifting it is force x distance = 20 x 5 = 100 joules, since lifting it at a steady speed requires a lifting force that just balances the weight. This 100 joules is now stored ready for use, that is, it is potential energy. Upon releasing the hammer, the potential energy becomes kinetic energy—the force of gravity pulls the hammer downwards through the same distance the hammer was originally raised upwards, so since it's a force of the same size as the original lifting force, the work done on the hammer by gravity in giving it motion is the same as the work done previously in lifting it, so as it hits the nail it has a kinetic energy of 100 joules. We say that the potential energy is transformed into kinetic energy, which is then spent driving in the nail.

We should emphasize that both energy and work are measured in the same units, joules. In the example above, doing work by lifting just adds energy to a body, so-called potential energy, equal to the amount of work done.

From the above discussion, a mass of m kilograms has a weight of mg newtons. It follows that the work needed to raise it through a height h meters is force x distance, that is, weight x height, or mgh joules. This is the potential energy.

Historically, this was the way energy was stored to drive clocks. Large weights were raised once a week and as they gradually fell, the released energy turned the wheels and, by a sequence of ingenious devices, kept the pendulum swinging. The problem was that this necessitated rather large clocks to get a sufficient vertical drop to store enough energy, so spring-driven clocks became more popular when they were developed. A compressed spring is just another way of storing energy. It takes work to compress a spring, but (apart from small frictional effects) all that work is released as the spring uncoils or springs back. The stored energy in the compressed spring is often called *elastic potential energy*, as opposed to the *gravitational potential energy* of the raised weight.

Kinetic Energy

We've given above an explicit way to find the potential energy increase of a mass m when it's lifted through a height h, it's just the work done by the force that raised it, force x distance = weight x height = mgh.

Kinetic energy is created when a force does work accelerating a mass and increases its speed. Just as for potential energy, we can find the kinetic energy created by figuring out how much work the force does in speeding up the body.

Remember that a force only does work if the body the force is acting on moves in the direction of the force. For example, for a satellite going in a circular orbit around the earth, the force of gravity is constantly accelerating the body downwards, but it never gets any closer to sea level, it just swings around. Thus the body does not actually move any distance in the direction gravity's pulling it, and in this case gravity does no work on the body.

Consider, in contrast, the work the force of gravity does on a stone that is simply dropped from a cliff. Let's be specific and suppose it's a one kilogram stone, so the force of gravity is ten newtons downwards. In one second, the stone will be moving at ten meters per second, and will have dropped five meters. The work done at this point by gravity is force x distance = 10 newtons x 5 meters = 50 joules, so this is the kinetic energy of a one kilogram mass going at 10 meters per second. How does the kinetic energy increase with speed? Think about the situation after 2 seconds. The mass has now increased in speed to twenty meters per second. It has fallen a total distance of twenty meters (average speed 10 meters per second x time elapsed of 2 seconds). So the work done by the force of gravity in accelerating the mass over the first two seconds is force x distance = 10 newtons x 20 meters = 200 joules.

So we find that the kinetic energy of a one kilogram mass moving at 10 meters per second is 50 joules, moving at 20 meters per second it's 200 joules. It's not difficult to check that after three seconds, when the mass is moving at 30 meters per second, the kinetic energy is 450 joules. The essential point is that the speed increases linearly with time, but the work done by the constant gravitational force depends on how far the stone has dropped, and that goes as the square of the time. Therefore, the kinetic energy of the falling stone depends on the square of the time, and that's the same as depending on the square of the velocity. For stones of different masses, the kinetic energy at the same speed will be proportional to the mass (since weight is proportional to mass, and the work done by gravity is proportional to the weight), so using the figures we worked out above for a one kilogram mass, we can conclude that for a mass of *m* kilograms moving at a speed *v* the kinetic energy must be:

$$\textbf{kinetic energy} = \tfrac{1}{2}mv^2$$

Exercises for the reader: both momentum and kinetic energy are in some sense measures of the amount of motion of a body. How do they differ?

Can a body change in momentum without changing in kinetic energy?

Can a body change in kinetic energy without changing in momentum?

Suppose two lumps of clay of equal mass traveling in opposite directions at the same speed collide head-on and stick to each other. Is momentum conserved? Is kinetic energy conserved?

As a stone drops off a cliff, both its potential energy and its kinetic energy continuously change. How are these changes related to each other?

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