More Rotational Dynamics

Physics 1425 Lecture 20

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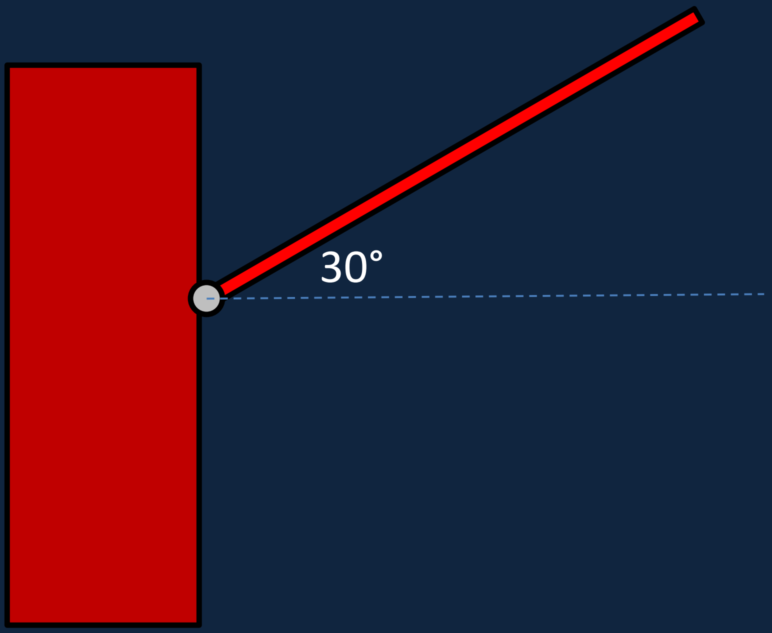
Clicker Question

A uniform rod is free to rotate in a vertical plane about a frictionless hinge at one end. It is released from rest at an angle of 30°. ($I = (1/3)ML^2$, $\tau = Mg(L/2)\cos 30°$) The initial downward acceleration of the free end of the rod is:



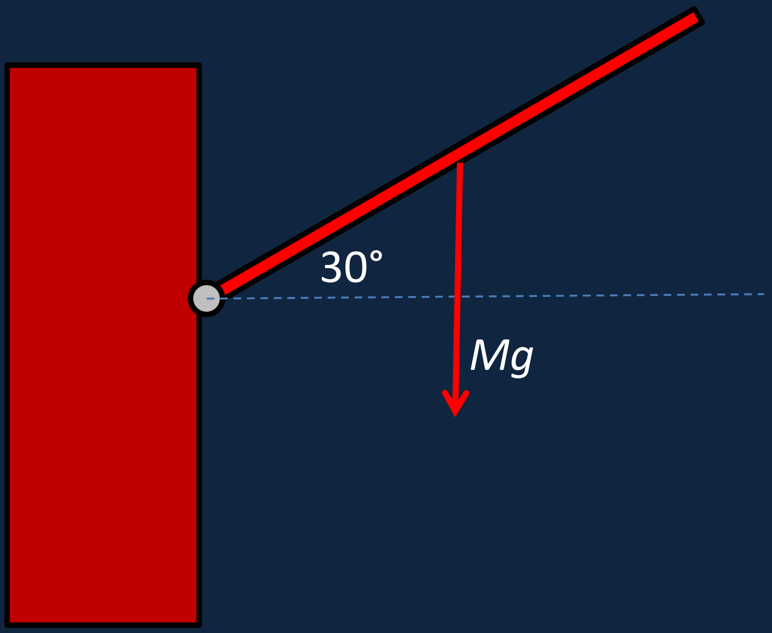
A. equal to g

B. greater than g

C. less than g

Clicker Answer

It's *greater* than g! The moment of inertia about the hinge is $(1/3)ML^2$, the torque is $(MgL/2)\cos 30°$, so the acceleration is given by $\tau = I\alpha$, $\alpha = (3g/2L)\cos 30°$, the far end accelerates at $L\alpha = (3g/2)\cos 30° > g$.



Falling coins

Rotational Kinetic Energy

- Imagine a rotating body as composed of many small masses m_i at distances r_i from the axis of rotation.
- The mass m_i has speed $v = \omega r_i$, so $KE = ½m_i r_i^2 \omega^2$.
- The total *KE* of the rotating body (assuming the axis is at rest) is

$$K = \sum_i \left(\tfrac{1}{2} m_i r_i^2\right)\omega^2 = \tfrac{1}{2} I \omega^2$$

Torque Power

- If a net torque τ is acting on a rotating body, the net power is the rate of change of rotational energy

$$\frac{d}{dt}\left(\tfrac{1}{2}I\omega^2\right) = I\omega\frac{d\omega}{dt} = I\omega\alpha = \omega\tau \;\; (\text{recall } \; \tau = I\alpha)$$

- So the rate of working of the torque,

  power = $\tau\omega$, its value x the angular velocity.
- Total work done over some time period is

$$\int \tau\omega dt = \int \tau \frac{d\theta}{dt} dt = \int \tau d\theta$$

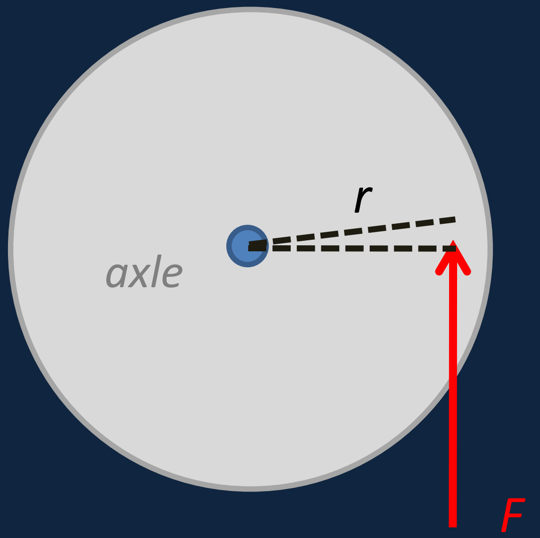
- This is just like $\int F dx$ in linear motion.

Work Done by a Torque

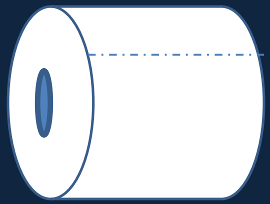
- Suppose the torque is a force F acting at a distance r from the center as shown. If the disk turns through an angle $d\theta$, the force acts through a distance $ds = rd\theta$ so does work $Fds = Frd\theta$.
- But $\tau = rF$, so the work

$$Fds = Frd\theta = \tau d\theta$$

Force x distance = torque x angle



A Familiar Item…



- A roll of toilet paper has diameter 0.1m, which happens also to be the length of one sheet.
- What is the angle *in radians* subtended at the central line of the roll by one sheet in the outside layer?
- A. 1

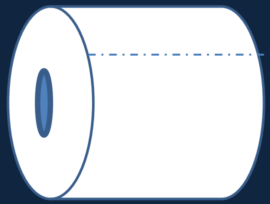
  B. 2

  C. 0.5

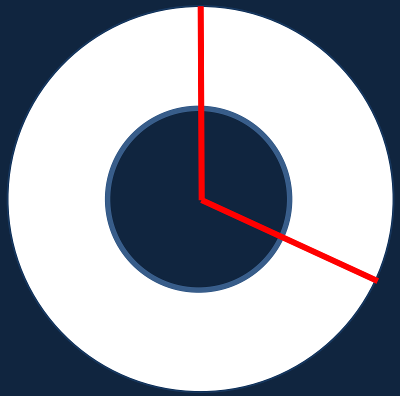
  D. π

  E. 1/π

A Familiar Item…



- A roll of toilet paper has diameter 0.1m, which happens also to be the length of one sheet.
- What is the angle *in radians* subtended at the central line of the roll by one sheet in the outside layer?
- It's about 2 radians:



On a Roll…

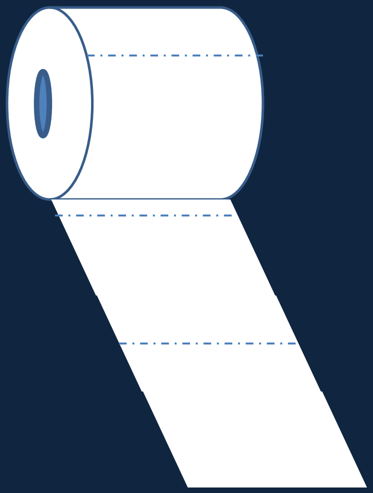
- This roll (0.1 m diameter, 0.1 m sheets) rolls across the table, unwinding three sheets per second.
- Give its CM velocity, *and* the angular velocity about the CM in radians/sec.

A. 0.3, 6

B. 0.3, 3

C. 0.6, 6

D. 0.3, 3π



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A. 0.3, 6

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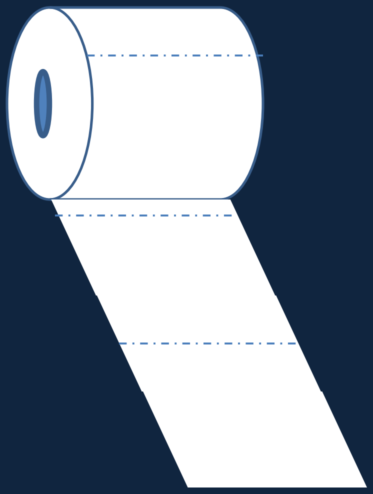
C. 0.6, 6

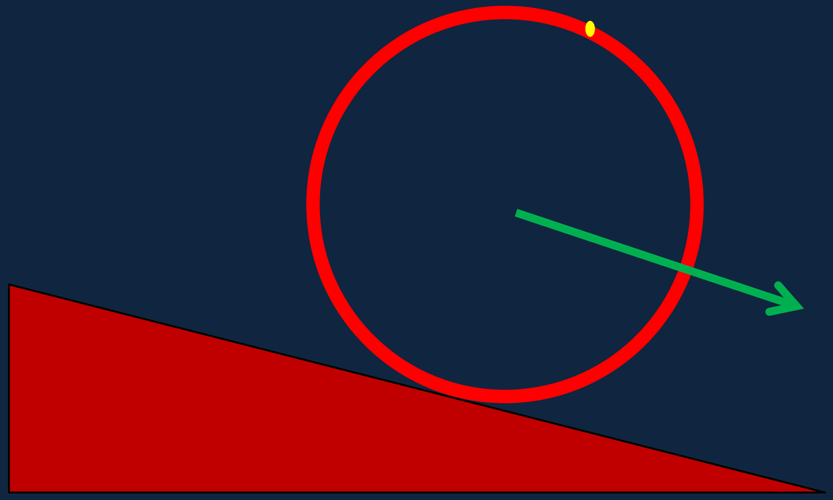
D. 0.3, 3π

Remember $\omega = vr$, and three sheets in one second is 6 radians—almost a complete revolution.



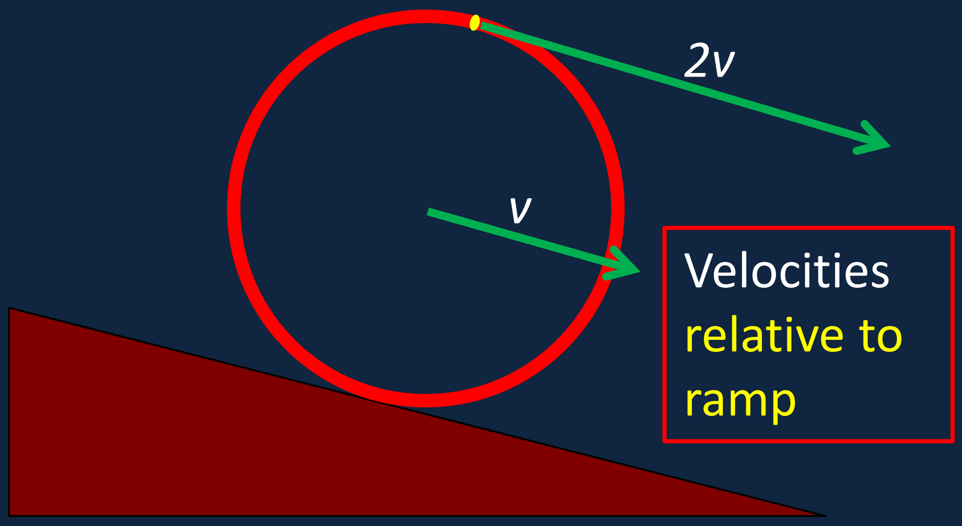
Clicker Question

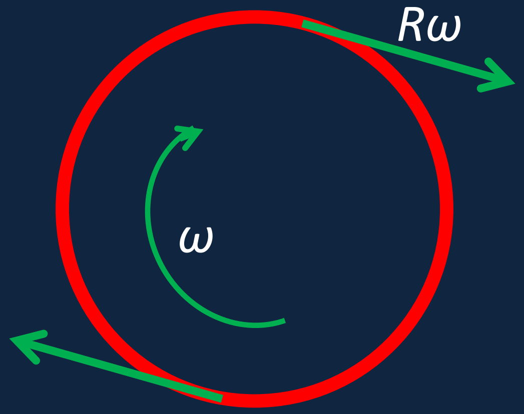
- A hoop is rolling down a ramp (without slipping) at *v* m/sec.
- How fast is the point on the hoop furthest from the ramp moving?
- A. *v* m/sec
- B. 2*v* m/sec
- C. 4*v* m/sec



Hoop Rolling Down Ramp

- If there's no slipping, the point on the hoop in contact with the ramp is at *rest*—*the hoop is at that instant rotating about that point.*
- So if the center is moving at v, the "top" point is moving at $2v$.
- Relative to the center, all points are moving at speed $R\omega$ tangentially.
- Hence, since the bottom's at rest: $v = R\omega$
- The "<u>no slip</u>" condition.





Velocities relative to center of hoop

Total Kinetic Energy of Rolling Hoop

- Suppose as usual the hoop is made of many small masses m_i and the mass m_i is moving at $\vec{v}_i$. Then the total KE is $\sum_i \frac{1}{2} m_i \vec{v}_i^{\,2}$.
- This total kinetic energy depends on both the translational motion (the center of the hoop is moving) and the hoop's rotation about the center.
- How do we sort this out?

Separating Translational and Rotational Kinetic Energies: Details

- Suppose we have rigid body we represent as a collection of masses m_i, with individual velocities $\vec{v}_i$.
- Let's suppose the CM is moving at $\vec{v}_{\mathrm{CM}}$, so the total linear momentum is $M\vec{v}_{\mathrm{CM}}$, M being the total mass.
- To separate out the rotational motion, we'll write the individual velocities $\vec{v}_i = \vec{v}_{\mathrm{CM}} + \vec{u}_i$: so $\vec{u}_i$ is velocity of m_i relative to the CM.
- Then the total kinetic energy is

$$\sum_i \tfrac{1}{2} m_i \vec{v}_i^2 = \sum_i \tfrac{1}{2} m_i \left(\vec{v}_{\mathrm{CM}} + \vec{u}_i\right)^2 = \tfrac{1}{2} M\vec{v}_{\mathrm{CM}}^2 + \vec{v}_{\mathrm{CM}} \cdot \sum_i m_i \vec{u}_i + \sum_i \tfrac{1}{2} m_i \vec{u}_i^2$$

$$KE = \tfrac{1}{2} M\vec{v}_{\mathrm{CM}}^2 + \tfrac{1}{2} I_{\mathrm{CM}} \omega^2$$

- Because relative to the CM $\sum_i m_i \vec{u}_i = \dfrac{d}{dt} \sum_i m_i \vec{r}_i = 0, \quad \vec{u}_i^2 = r_i^2 \omega^2.$

Total Energy: the Bottom Line

- In case the last slide was too much, what you *really* need is that the total kinetic energy of a moving, rotating object is a sum of two terms:
- Translational KE, the same as if all the mass is moving with the velocity of the center of mass, and
- Rotational KE, about the center of mass:

$$KE = \frac{1}{2} M \vec{v}_{\mathrm{CM}}^2 + \frac{1}{2} I_{\mathrm{CM}} \omega^2$$

How Fast Does a Hoop Roll Down a Ramp?

- Assuming no slipping, so

$$v = R\omega$$

- The total kinetic energy at an instant:

$KE = ½mv^2 + ½I\omega^2$

$= ½mv^2 + ½(mR^2)\omega^2$

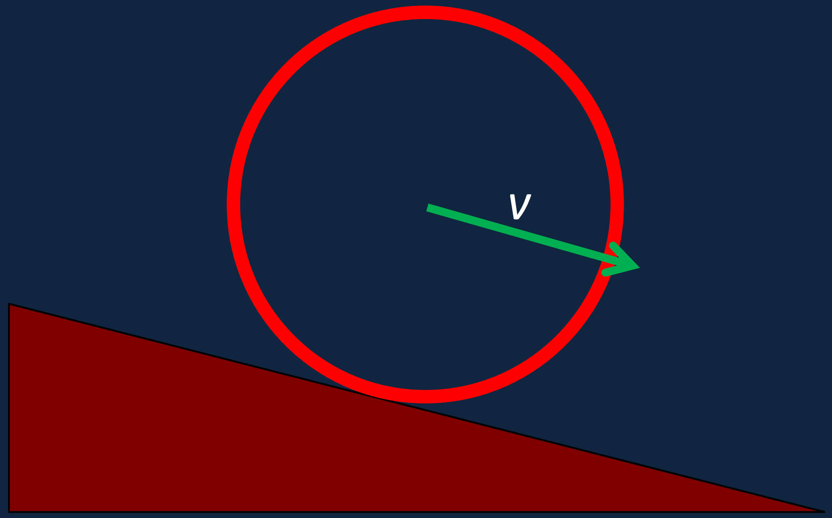
$= mv^2$.

- If it's rolled down through height h from a standing start,

$mv^2 = mgh$, so $v = \sqrt{(gh)}$

- For a frictionless sliding mass,

$½mv^2 = mgh$, so $v = \sqrt{(2gh)}$: faster!



The hoop takes longer to get down than a low-friction sliding block, because the same loss in potential energy has to supply BOTH translational *KE* and rotational *KE* for the hoop.

Ramp Race

A hoop, a solid cylinder and a solid sphere roll down the same ramp from a standing start. Who clocks the fastest time?

A. The hoop
B. The solid cylinder
C. The solid sphere
D. It depends on the sizes and/or masses.

Ramp Race

A hoop, a solid cylinder and a solid sphere roll down the same ramp from a standing start. Who clocks the fastest time?

The sphere wins: its mass is on average closer to the axis of rotation, so it has less rotational *KE* compared with translational *KE*.

A. The hoop
B. The solid cylinder
C. The solid sphere
D. It depends on the sizes and/or masses.

Note: for the sphere $I = (2/5)mR^2$ solid cylinder $\frac{1}{2}mR^2$, hoop mR^2.

A New Look for $\tau = I\alpha$

- We've seen how $\tau = I\alpha$ works for a body rotating about a fixed axis.
- $\tau = I\alpha$ *i*s not true in general if the axis of rotation is *itself* accelerating
- BUT it IS true if the axis is through the CM, and isn't changing direction!
- This is quite tricky to prove—it's in the book
- And $\boxed{\tau_{CM} = I_{CM}\alpha_{CM}}$ is often useful, as we'll see.