

# Angular Momentum

## Physics 1425 Lecture 21

# A New Look for $\tau = I\alpha$

- We've seen how  $\tau = I\alpha$  works for a body rotating about a **fixed axis**.
- $\tau = I\alpha$  is not true in general if the axis of rotation is *itself* accelerating
- **BUT it IS true if the axis is through the CM, and isn't changing direction!**
- This is quite tricky to prove—it's in the book
- And  $\tau_{\text{CM}} = I_{\text{CM}}\alpha_{\text{CM}}$  is often useful, as we'll see.

# Forces on Hoop Rolling Down Ramp

- Take no slipping, so

$$v = R\omega, \quad a = R\alpha$$

- Translational accn  $F = ma$ :

$$mg\sin\theta - F_{fr} = ma$$

- Rotational accn  $\tau_{CM} = I_{CM}\alpha_{CM}$ :

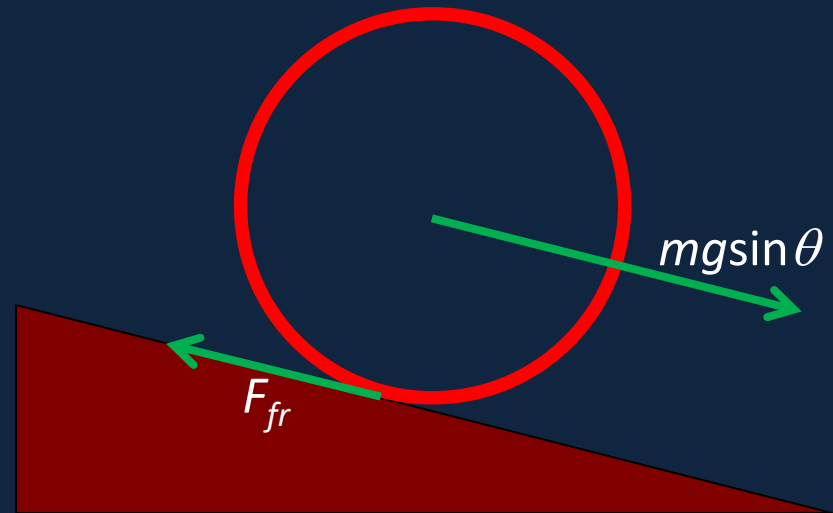
$$F_{fr}R = mR^2\alpha = mRa$$

so  $F_{fr} = ma$  and

$$mg\sin\theta = 2ma,$$

- $a = (g\sin\theta)/2$ :

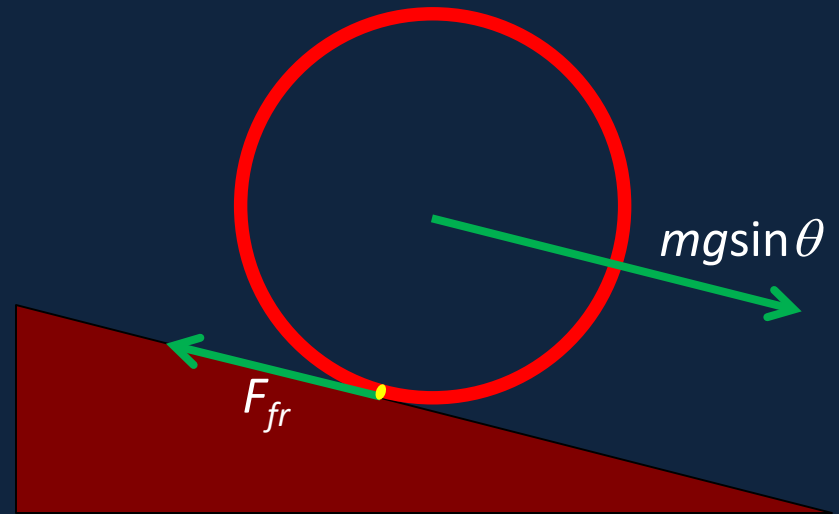
the acceleration is **one-half** that of a sliding frictionless block—and independent of mass or radius.



The **only** force having torque about the center of the hoop (its CM) is the **frictional force**: the total gravitational force and the normal force both act through the center.

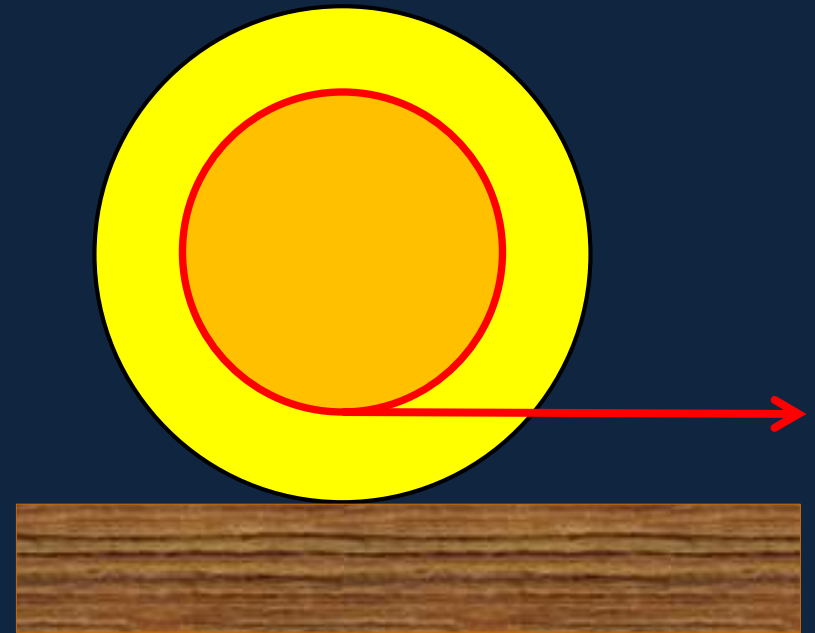
# Yet Another Look at That Hoop...

- Take no slipping, so
$$v = R\omega, \quad a = R\alpha$$
- Since there's no slipping, the point on the hoop in contact with the ramp is momentarily at rest, and the hoop is rotating about that point.
- The only torque about that point is gravity— $\tau = mgR\sin\theta$
- The moment of inertia about that point, from the parallel axis theorem, is  $I_{\text{CM}} + mR^2 = 2mR^2$ , so  $mgR\sin\theta = 2mR^2\alpha$ , and  $a = \alpha/R = (g\sin\theta)/2$ .



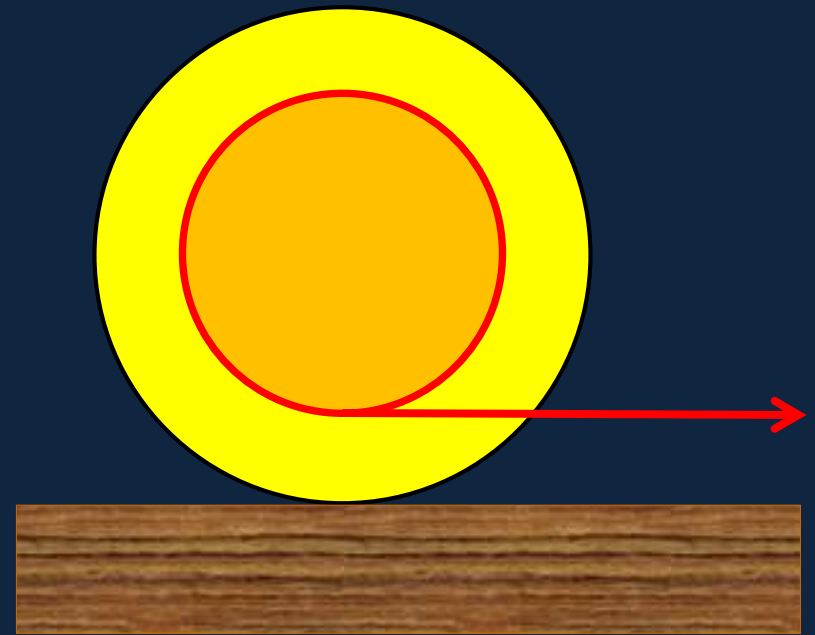
# Clicker Question

- A wooden yo-yo with red string rests on a table top. I pull the string horizontally from the bottom. What will the yo-yo do? (Assume ordinary smooth wood.)
- A. Roll towards me.
  - B. Roll away from me.
  - C. Slide towards me.



# Clicker Answer

- A wooden yo-yo with red string rests on a table top. I pull the string horizontally from the bottom. What will the yo-yo do? (Assume ordinary smooth wood.)



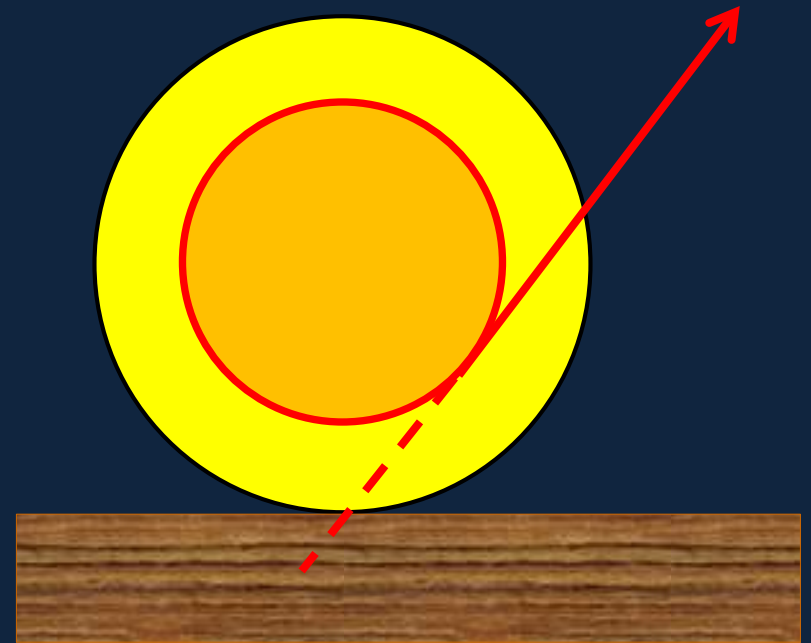
- A. Roll towards me.
- B. Roll away from me.
- C. Slide towards me.

The key is to measure torque about the stationary point of contact of the yo-yo with the table. Clearly the torque is clockwise!

# Clicker Question

- A wooden yo-yo with red string rests on a table top. I pull the string **along a line that passes through the point of contact**. What will the yo-yo do? (Assume ordinary smooth wood.)

- A. Roll towards me.
- B. Roll away from me.
- C. Slide towards me.



# Varying Moment of Inertia

- Recall Newton wrote his Second Law  $F = dp/dt$ , allowing  $m$  to vary as well as  $v$ .
- We should write the rotational version
- $\tau = d(l\omega)/dt$ , and in fact varying  $l$ 's are far more common than varying  $m$ 's.





# Clicker Question

- Assume that when she pulls herself inwards, the angular velocity increases by a factor of 3.
- What happens to 1: **total angular momentum** and 2: **rotational kinetic energy**?
  - A. No change, no change
  - B. No change, x3 increase.
  - C. x3 increase, x3 increase
  - D. x3 increase, x9 increase



# Torque as a Vector

- Suppose we have a wheel spinning about a fixed axis: then  $\vec{\omega}$  always points along the axis—so  $d\vec{\omega}/dt$  points along the axis too.

- If we want to write a vector equation

$$\vec{\tau} = I\vec{\alpha} = Id\vec{\omega}/dt$$

it's clear that the vector  $\vec{\tau}$  is parallel to the vector  $d\vec{\omega}/dt$ : so  $\vec{\tau}$  points along the axis too!

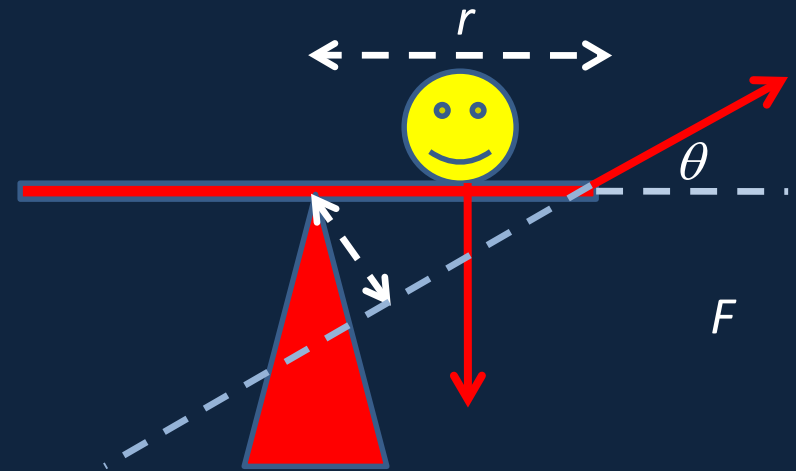
- **BUT** this vector  $\vec{\tau}$ , is, remember made of two other vectors: the force  $\vec{F}$  and the place  $\vec{r}$  where it acts!

# Recalling an Earlier Torque

- Only the component of  $F$  perpendicular to the arm exerts torque

$$\tau = rF \sin \theta$$

- We can see the direction of  $\vec{\tau}$  is perpendicular to both  $\vec{F}$ ,  $\vec{r}$  and towards us.
- We **define** the **vector cross product**  $\vec{\tau} = \vec{r} \times \vec{F}$  to have this direction, and magnitude  $rF \sin \theta$ .

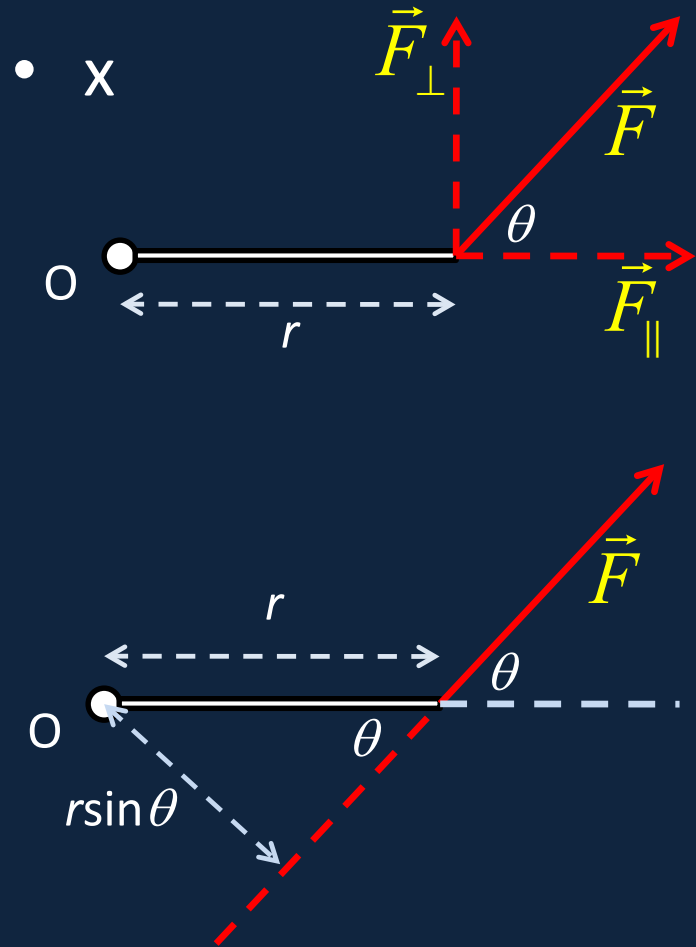


# More Torque...

- Expressing the force vector  $\vec{F}$  as a sum of components  $\vec{F}_\perp$  (“fperp”) perpendicular to the lever arm and  $\vec{F}_\parallel$  parallel to the arm, it’s clear that only  $\vec{F}_\perp$  has leverage, that is, torque, about O.

$\vec{F}_\perp$  has magnitude  $F\sin\theta$ , so  $\tau = rF\sin\theta$ .

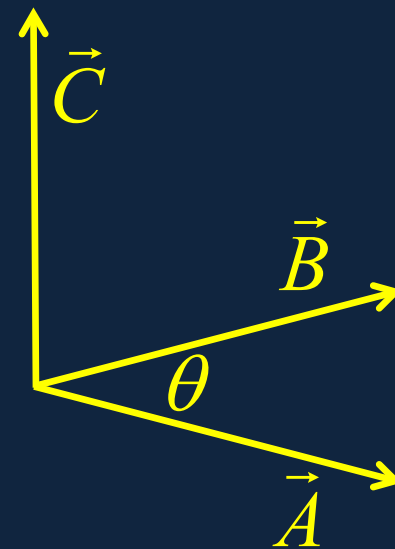
- Alternatively, keep  $\vec{F}$  and measure *its* lever arm about O: that’s  $r\sin\theta$ .



# Definition: The Vector Cross Product

$$\vec{C} = \vec{A} \times \vec{B}$$

- The **magnitude**  $C$  is  $AB\sin\theta$ , where  $\theta$  is the angle between the vectors  $\vec{A}, \vec{B}$ .
- The **direction** of  $\vec{C}$  is perpendicular to both  $\vec{A}$  and  $\vec{B}$ , and is your right thumb direction if your curling fingers go from  $\vec{A}$  to  $\vec{B}$ .



# Clicker Question

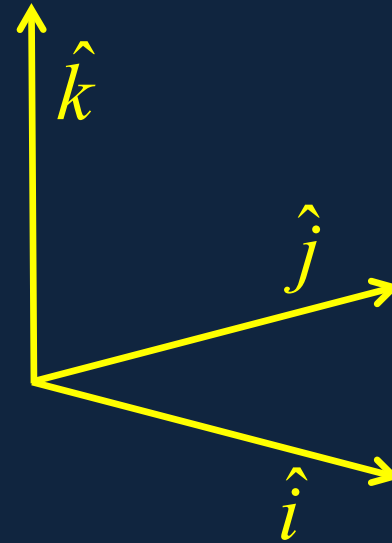
Assume  $\vec{A}, \vec{B}$  are **nonzero** vectors.

Which pair of statements below is correct?

- A. The cross product depends on the order of the factors, and since both vectors are nonzero, it can never be zero.
- B. Depends on order, can be zero.
- C. Doesn't depend on order, cannot be zero.
- D. Doesn't depend on order, can be zero.

# The Vector Cross Product in Components

- Recall we defined the unit vectors  $\hat{i}, \hat{j}, \hat{k}$  pointing along the x, y, z axes respectively, and a vector can be expressed as  $\vec{A} = A_x \hat{i} + A_y \hat{j} + A_z \hat{k}$



- Now  $\hat{i} \times \hat{i} = 0, \hat{i} \times \hat{j} = \hat{k}, \hat{i} \times \hat{k} = -\hat{j}, \dots$
- So

$$\begin{aligned}\vec{A} \times \vec{B} &= (A_x \hat{i} + A_y \hat{j} + A_z \hat{k}) \times (B_x \hat{i} + B_y \hat{j} + B_z \hat{k}) \\ &= \hat{i} (A_y B_z - A_z B_y) + \dots\end{aligned}$$