More Angular Momentum

Physics 1425 Lecture 22

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Torque as a Vector

- Suppose we have a wheel spinning about a fixed axis: then $\vec{\omega}$ always points along the axis—so $d\vec{\omega}/dt$ points along the axis too.
- If we want to write a vector equation

$$\vec{\tau} = I\vec{\alpha} = Id\vec{\omega}/dt$$

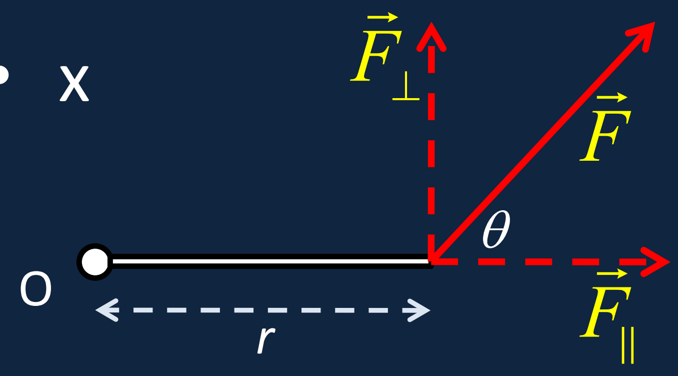
it's clear that the vector $\vec{\tau}$ is parallel to the vector $d\vec{\omega}/dt$: so $\vec{\tau}$ points along the axis too!

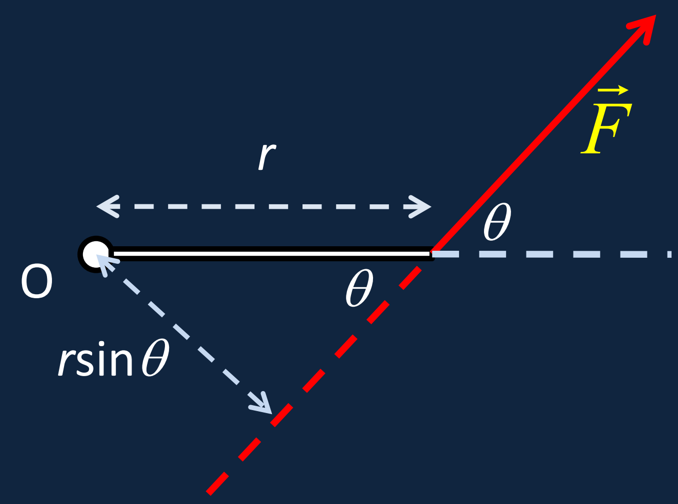
- BUT this vector $\vec{\tau}$, is, remember made of two other vectors: the force $\vec{F}$ and the place $\vec{r}$ where it acts!

More Torque…

- Expressing the force vector $\vec{F}$ as a sum of components $\vec{F}_\perp$ ("fperp") perpendicular to the lever arm and $\vec{F}_\parallel$ parallel to the arm, it's clear that only $\vec{F}_\perp$ has leverage, that is, torque, about O.
  $\vec{F}_\perp$ has magnitude $F\sin\theta$, so $\tau = rF\sin\theta$.
- Alternatively, keep $\vec{F}$ and measure *its* lever arm about O: that's $r\sin\theta$.

- x

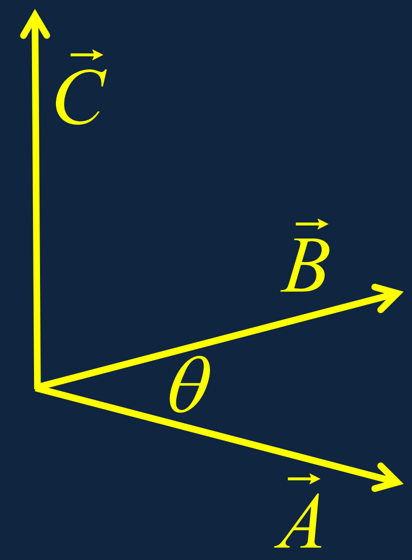




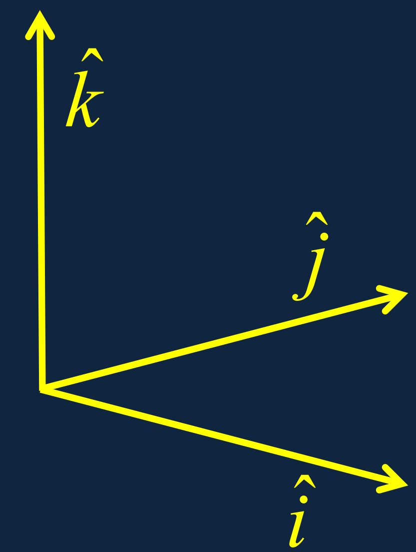
Definition: The Vector Cross Product

$$\vec{C} = \vec{A} \times \vec{B}$$

- The magnitude C is $AB\sin\theta$, where θ is the angle between the vectors $\vec{A}, \vec{B}$.
- The direction of $\vec{C}$ is perpendicular to both $\vec{A}$ and $\vec{B}$, and is your right thumb direction if your curling fingers go from $\vec{A}$ to $\vec{B}$.



The Vector Cross Product in Components



- Recall we defined the unit vectors $\hat{i}, \hat{j}, \hat{k}$ pointing along the *x*, *y*, *z* axes respectively, and a vector can be expressed as $\vec{A} = A_x\hat{i} + A_y\hat{j} + A_z\hat{k}$
- Now $\hat{i}\times\hat{i} = 0,\ \hat{i}\times\hat{j} = \hat{k},\ \hat{i}\times\hat{k} = -\hat{j},\ldots$
- So

$$\vec{A}\times\vec{B} = \left(A_x\hat{i} + A_y\hat{j} + A_z\hat{k}\right)\times\left(B_x\hat{i} + B_y\hat{j} + B_z\hat{k}\right)$$

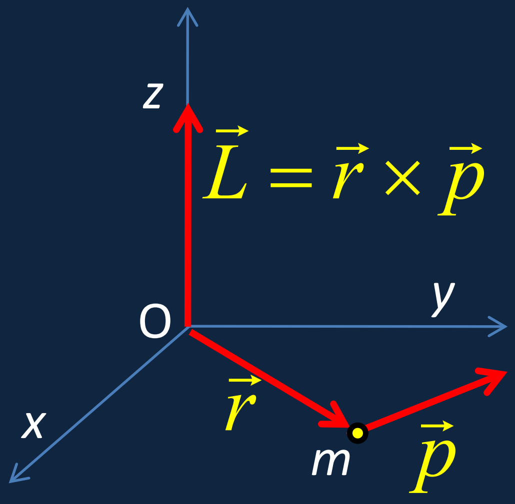
$$= \hat{i}\left(A_yB_z - A_zB_y\right) + \ldots$$

Vector Angular Momentum of a Particle

- A particle with momentum $\vec{p}$ is at position $\vec{r}$ from the origin O.
- Its angular momentum about the origin is

$$\vec{L} = \vec{r} \times \vec{p}$$

- This is in line with our definition for part of a rigid body rotating about an axis: *but also works for a particle flying through space*.



Viewing the *x*-axis as coming out of the slide, this is a “right-handed” set of axes: $\hat{i} \times \hat{j} = +\hat{k}$

Angular Momentum and Torque for a Particle

- Angular momentum about the origin of particle mass *m*, momentum $\vec{p}$ at $\vec{r}$

$$\vec{L} = \vec{r} \times \vec{p}$$

- Rate of change:

$$\frac{d\vec{L}}{dt} = \frac{d\vec{r}}{dt} \times \vec{p} + \vec{r} \times \frac{d\vec{p}}{dt} = \vec{r} \times \vec{F} = \vec{\tau}$$

Torque about the origin

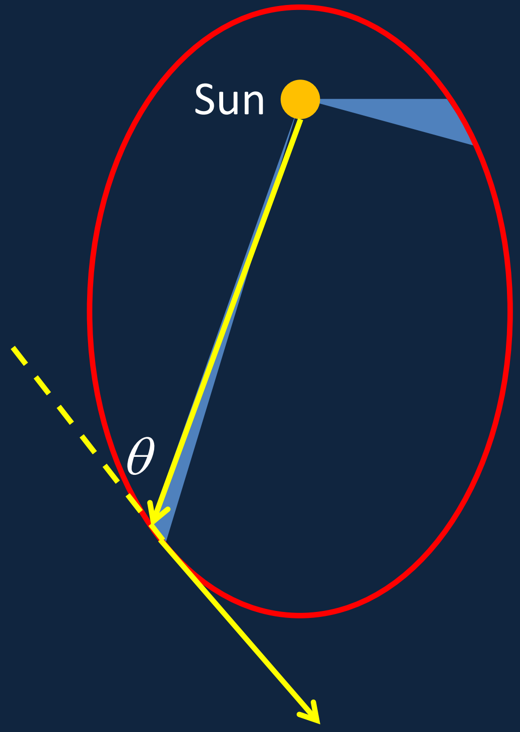
- because

$$\frac{d\vec{r}}{dt} \times \vec{p} = \vec{v} \times m\vec{v} = 0.$$

Kepler's Second Law

As the planet moves, a line from the planet to the center of the Sun sweeps out equal areas in equal times.

- In unit time, it moves through a distance $\vec{v}$.
- The area of the triangle swept out is ½*rv*sinθ (from ½ base x height)
- This is ½*L*/*m*, $\vec{L} = \vec{r} \times \vec{p}$.
- Kepler's Law is telling us the angular momentum about the Sun is constant: this is because the Sun's pull has *zero torque* about the Sun itself.



The base of the thin blue triangle is a distance *v* along the tangent. The height is the perp distance of this tangent from the Sun.

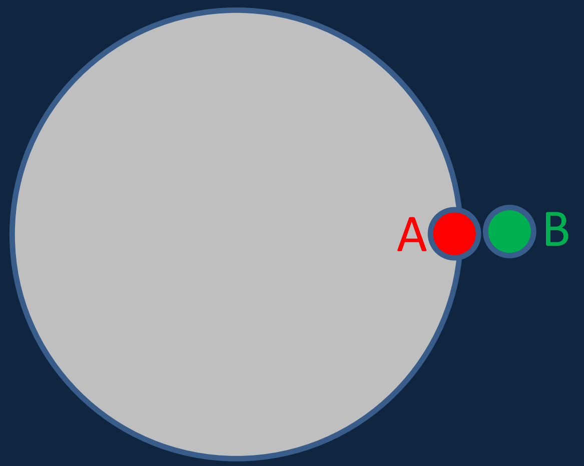
Guy on Turntable

- A, of mass m, is standing on the edge of a frictionless turntable, a disk of mass $4m$, radius R, next to B, who's on the ground.
- A now walks around the edge until he's back with B.
- How far does he walk?

A. $2\pi R$

B. $2.5\pi R$

C. $3\pi R$

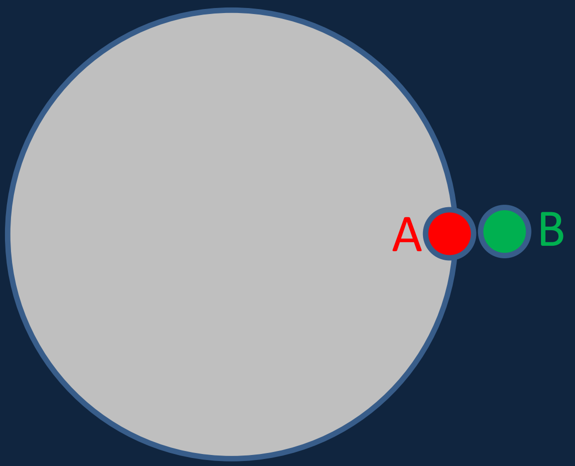


Guy on Turntable: Answer

- A, of mass m, is standing on the edge of a frictionless turntable, a disk of mass $4m$, radius R, next to B, who's on the ground.
- A now walks around the edge until he's back with B. How far does he walk?

  $3\pi R$

His moment of inertia is mR^2, the turntable's is $2mR^2$. There is zero total angular momentum, so if he walks around with angular velocity ω relative to the ground, the turntable has angular velocity $-\omega/2$. If he marked the turntable at the point he began, he'd reach that mark again after walking 2/3rds of the way round, as the turntable turned the other way to meet him. When he gets back to B, the turntable has done half a complete turn.



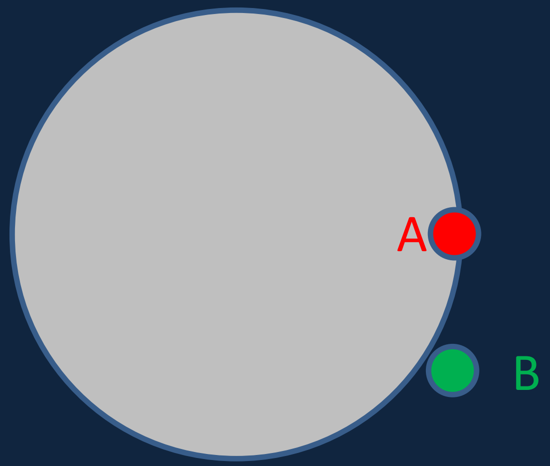
Guy on Turntable Catches a Ball

- A, of mass *m*, is standing on the edge of a frictionless turntable, a disk of mass 4*m*, radius *R*, at rest.
- B, who's on the ground, throws a ball weighing 0.1*m* at speed *v* to A, who catches it without slipping.
- What is the angular momentum of turntable + man + ball now?

A. 0.1mvR

B. (0.1/3.1)mvR

C. (0.1/5.1)mvR



On the Ball? Answer

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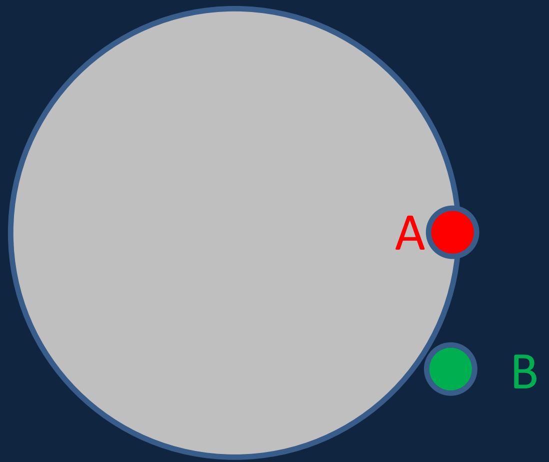


A. 0.1mvR
B. (0.1/3.1)mvR
C. (0.1/5.1)mvR

The ball thrown from B to A is moving in the direction of the tangent at A, the angular momentum about a point of a particle flying through the air equals $\vec{r} \times m\vec{v}$ and the line of the velocity is perp to the radius ending at A, so the angular momentum of the ball about the disk center is 0.1*mvR*. There is no other angular momentum, so this is shared with the man and the turntable.

Guy on Turntable Walks In

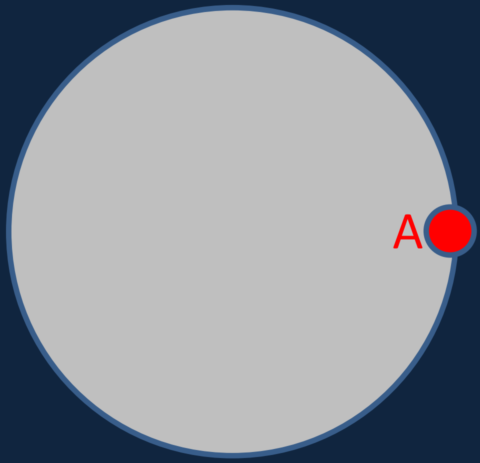
- A, of mass *m*, is standing on the edge of a frictionless turntable, a disk of mass 4*m*, radius *R*, which is rotating at 6 rpm.
- A walks to the exact center of the turntable.
- How fast (approximately) is the turntable now rotating?

A. 12 rpm

B. 9 rpm

C. 6 rpm

D. 4 rpm



Guy on Turntable Walks In: Answer

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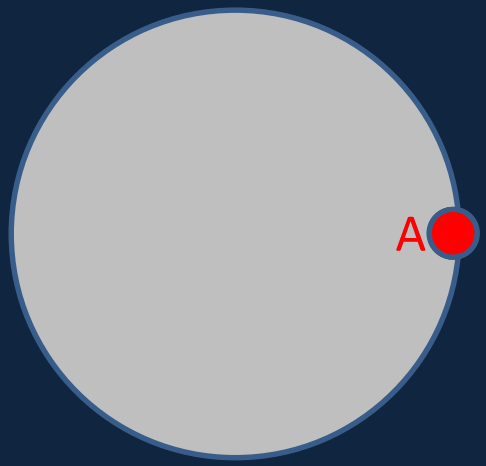


A. 12 rpm

B. 9 rpm

C. 6 rpm

D. 4 rpm

Initially, the man has moment of inertia mR^2, the turntable $2mR^2$. Finally, the man has negligible moment of inertia, so the total I decreases by a factor of 2/3, to conserve angular momentum (ther are no external torques) ω increases by 3/2.

Reminder: Angular Momentum and Torque for a Particle…

- Angular momentum about the origin of particle mass *m*, momentum $\vec{p}$ at $\vec{r}$

$$\vec{L} = \vec{r} \times \vec{p}$$

- Rate of change:

$$\frac{d\vec{L}}{dt} = \frac{d\vec{r}}{dt} \times \vec{p} + \vec{r} \times \frac{d\vec{p}}{dt} = \vec{r} \times \vec{F} = \vec{\tau}$$

- because

$$\frac{d\vec{r}}{dt} \times \vec{p} = \vec{v} \times m\vec{v} = 0.$$

Lots of Particles

- Suppose we have particles acted on by external forces, and also acting on each other.
- The rate of change of angular momentum of one of the particles about a fixed origin O is:

$$d\vec{L}_i / dt = \vec{\tau}_{i\ \mathrm{int}} + \vec{\tau}_{i\ \mathrm{ext}}$$

- The internal torques come in equal and opposite pairs, so

$$d\vec{L} / dt = \sum_i d\vec{L}_i / dt = \sum_i \vec{\tau}_{i\ \mathrm{ext}}$$

Rotational Motion of a Rigid Body

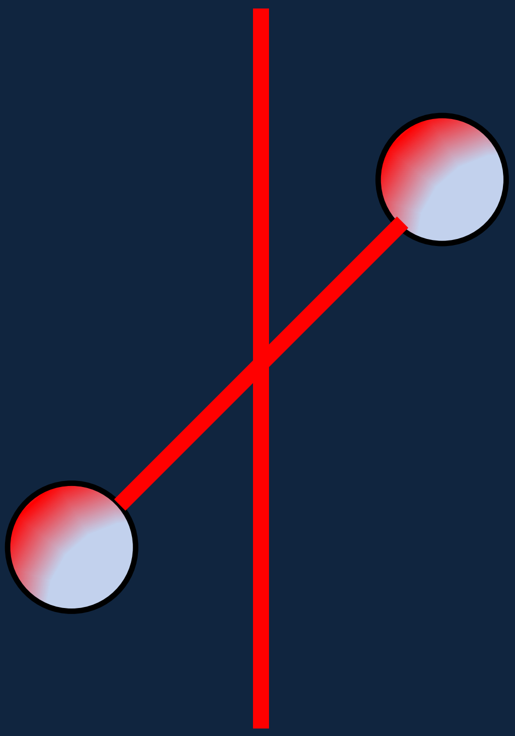
- For a collection of interacting particles, we've seen that

$$d\vec{L}/dt = \sum_i \vec{\tau}_i$$

  the vector sum of the applied torques, $\vec{L}$ and the $\vec{\tau}_i$ being measured about a fixed origin O.
- A rigid body is equivalent to a set of connected particles, so the same equation holds.
- It is also true (proof in book) that even if the CM is accelerating,

$$d\vec{L}_{\mathrm{CM}}/dt = \sum \vec{\tau}_{\mathrm{CM}}$$

Angular Velocity and Angular Momentum Need not be Parallel



- Imagine a dumbbell attached at its center of mass to a light vertical rod as shown, then the system rotates about the vertical line.
- The angular velocity vector $\vec{\omega}$ is vertical.
- The total angular momentum $\vec{L}$ about the CM is $\vec{r}_1 \times m\vec{v}_1 + \vec{r}_2 \times m\vec{v}_2$.
- Think about this at the instant the balls are in the plane of the slide—so is $\vec{L}$, but it's not vertical!

When *are* Angular Velocity and Angular Momentum Parallel?

- When the rotating object is symmetric about the axis of rotation: if for each mass on one side of the axis, there's an equal mass at the corresponding point on the other side.
- For this pair of masses, $\vec{r}_1 \times m\vec{v}_1 + \vec{r}_2 \times m\vec{v}_2$ *is* along the axis.
- (Check it out!)