

# More Angular Momentum, then Statics

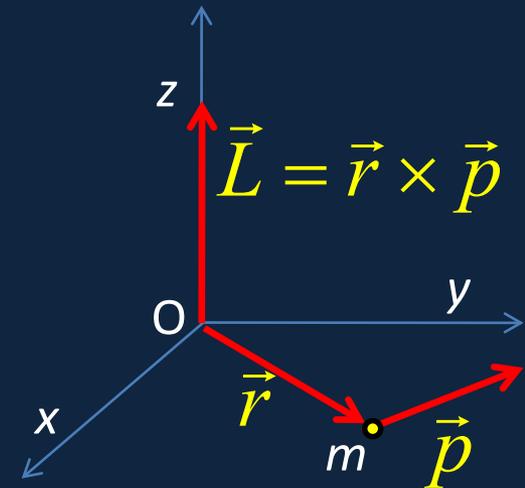
Physics 1425 Lecture 23

# Vector Angular Momentum of a Particle

- A particle with momentum  $\vec{p}$  is at position  $\vec{r}$  from the origin O.
- Its angular momentum about the origin is

$$\vec{L} = \vec{r} \times \vec{p}$$

- This is in line with our definition for part of a rigid body rotating about an axis: *but also works for a particle flying through space.*



Viewing the x-axis as coming out of the slide, this is a “right-handed” set of axes:

$$\hat{i} \times \hat{j} = +\hat{k}$$

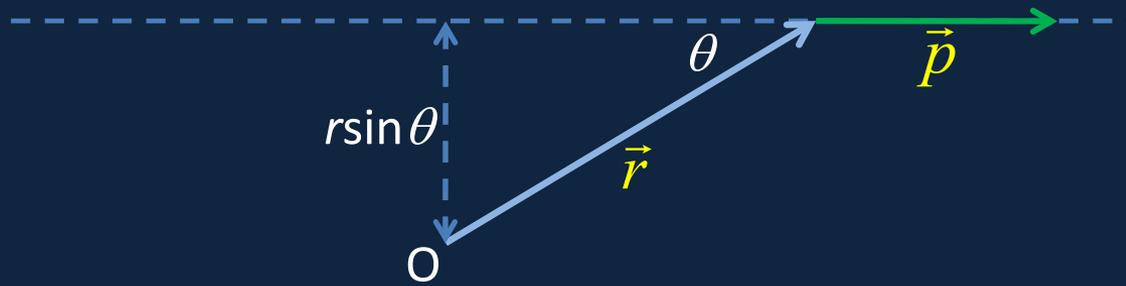
# Clicker Question

- A particle moves along a straight line at constant speed. The line does not pass through the origin. Is the particle's angular momentum about the origin constant?
  - A. Yes
  - B. No

# Clicker Answer

- A particle moves along a straight line at constant speed. The line does not pass through the origin. Is the particle's angular momentum about the origin constant?

A. Yes:  $|\vec{L}| = |\vec{r} \times \vec{p}| = (r \sin \theta) p$  and  $r \sin \theta$  is just the perpendicular distance of the line of motion from the origin—this is the same for any point on the line.



# Rotational Motion of a Rigid Body

- For a collection of interacting particles, we've seen that

$$d\vec{L} / dt = \sum_i \vec{\tau}_i$$

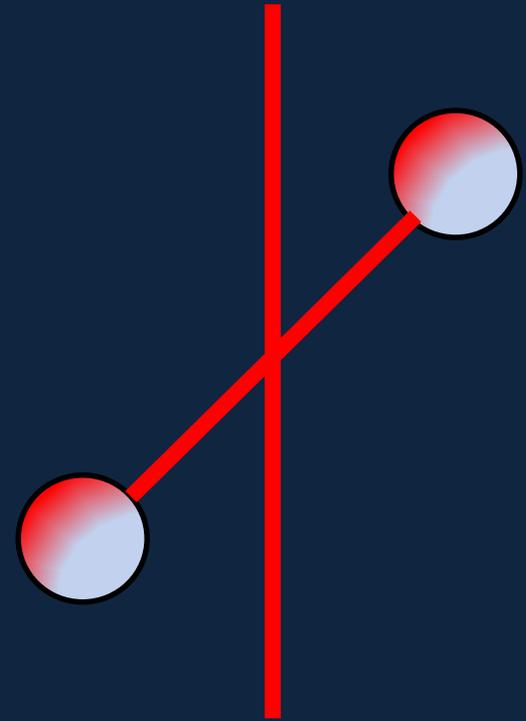
the vector sum of the applied torques,  $\vec{L}$  and the  $\vec{\tau}_i$ , being measured about a fixed origin O.

- A rigid body is equivalent to a set of connected particles, so the same equation holds.
- It is also true (proof in book) that even if the CM is accelerating,

$$d\vec{L}_{\text{CM}} / dt = \sum \vec{\tau}_{\text{CM}}$$

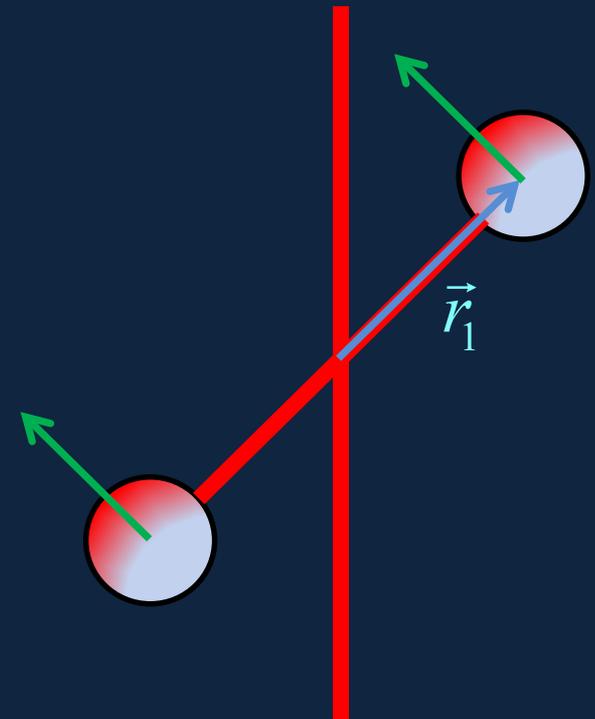
A dumbbell (two small masses at the ends of a light rigid rod) is mounted on a fixed axle through its center, at an angle  $\theta$ . It is set in steady rotation. The direction of the angular momentum of the system is:

- A. Along the axle
- B. Along the dumbbell rod
- C. Neither of the above.



# Clicker Answer

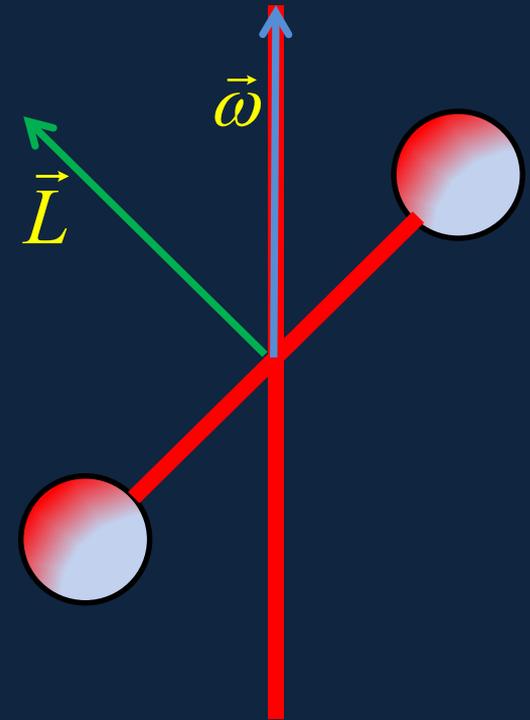
- The angular velocity vector  $\vec{\omega}$  is vertical.
- The total angular momentum  $\vec{L}$  about the CM is  $\vec{r}_1 \times m\vec{v}_1 + \vec{r}_2 \times m\vec{v}_2$ .
- Assume we're looking at the rotation at the instant when the ball to the right is moving **directly inwards** (into the screen). Then  $\vec{r}_1 \times m\vec{v}_1$  is in the plane of the screen, as shown. BUT  $\vec{r}_2 \times m\vec{v}_2$  is **in the same direction!** So  $\vec{\omega}$  is constant, but  $\vec{L}$  is rotating.
- This means **the axle is supplying a rotating torque to the dumbbell**. This is what causes problems with unbalanced tires.



Wait a minute— isn't  $\vec{L} = I\vec{\omega}$  ?  
See next slide!

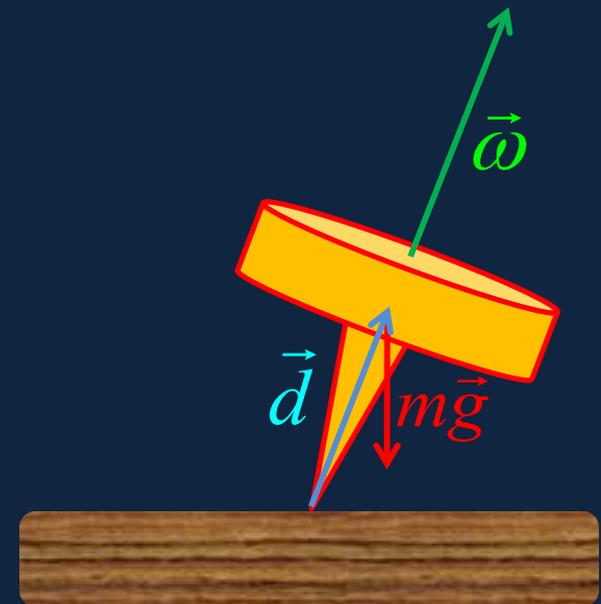
# A Bit More About $\vec{L}$ and $\vec{\omega}$ ...

- We've used  $\vec{L} = I\vec{\omega}$  a lot.
- We see from this example it's not always true that  $\vec{L}, \vec{\omega}$  are parallel vectors.
- What's going on?
- The answer is that  $\vec{L}, \vec{\omega}$  are only parallel if the spinning body is symmetric about the axis of rotation—which is usually the case.
- For more complicated cases, you will still see  $\vec{L} = \mathbf{I}\vec{\omega}$ , but that fat  $\mathbf{I}$  denotes a tensor or matrix.



# Spinning Top

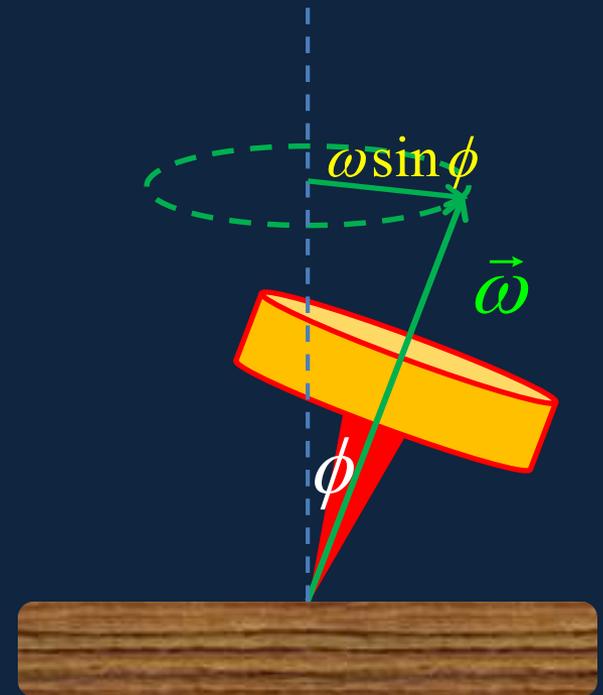
- Pointing your right thumb in the direction of the angular velocity vector  $\vec{\omega}$ , your curling fingers point in the direction of rotation.
- Gravity exerts a torque about the pivot point  $\vec{\tau} = \vec{d} \times m\vec{g}$ , evidently directed inwards.
- From  $\vec{\tau} = d\vec{L} / dt = Id\vec{\omega} / dt$   $d\vec{\omega}$  will be inwards, the tip of  $\vec{\omega}$  is describing a horizontal circle: this is “precession”.



# Precession Rate

$$\vec{\tau} = d\vec{L} / dt = I d\vec{\omega} / dt$$

- The horizontal component of the angular velocity vector  $\vec{\omega}$  has length  $\omega \sin \phi$  and it precesses around a circle centered above the pivot point.
- The precession angular velocity is written  $\Omega = d\theta / dt$ , where  $\theta$  measures angle around the horizontal circle.
- If in time  $dt$  there is precession through  $d\theta$ ,  $d\omega = (\omega \sin \phi) d\theta$
- so 
$$\frac{d\theta}{dt} = \frac{1}{\omega \sin \phi} \frac{d\omega}{dt} = \frac{1}{\omega \sin \phi} \frac{\tau}{I} = \frac{mgd}{\omega I}$$



# Statics: Conditions for Equilibrium

- For any body,  $Md\vec{v}_{\text{CM}} / dt = \sum \vec{F}_i$ , the net force causes the CM to accelerate. Hence, if the body is remaining at rest,

$$\sum_i \vec{F}_i = 0$$

- To eliminate *angular* acceleration, there must be zero torque about any axis. If all forces are in one plane, it's enough to prove zero torque about one axis perpendicular to the plane:

$$\sum_i \vec{\tau}_i = 0$$

# Free Body Diagrams

- To apply Newton's Laws to find how a body moves, we must focus on **that body alone** and add **all** the (vector) forces acting on it.
- The diagram showing all the forces on one body (or even part of a body) is called a "**free body diagram**"—we've "freed" the body from the rest of the system, representing everything else just by **the forces on this body**.
- The **net (total) force** then goes into  $\Sigma \vec{F} = m\vec{a}$ .

# Flat Forces?

- If a body in equilibrium is acted on by three and only three forces, do the force vectors have to lie in a plane?
  - A. Yes
  - B. No

# Flat Forces

- If just three forces are acting on a body, and it's in equilibrium, they must all lie in the same plane, because if we choose the plane defined by two of them, and the third force has a component perpendicular to that plane, nothing is balancing this perpendicular force.

# Clicker Question

- A body is in equilibrium. It is acted on by three forces, lying in a plane.
  - Do the lines of action of the three forces all pass through the same point?
- A. Yes
- B. No

# Three Force Equilibrium

- If a body is in equilibrium when acted on by three forces, the three forces must lie in the same plane AND all pass through a common point. If they don't, taking a perpendicular axis through a point where two of them meet, the third force gives an unbalanced torque about that point, so the body will have angular acceleration.

# Clicker Question

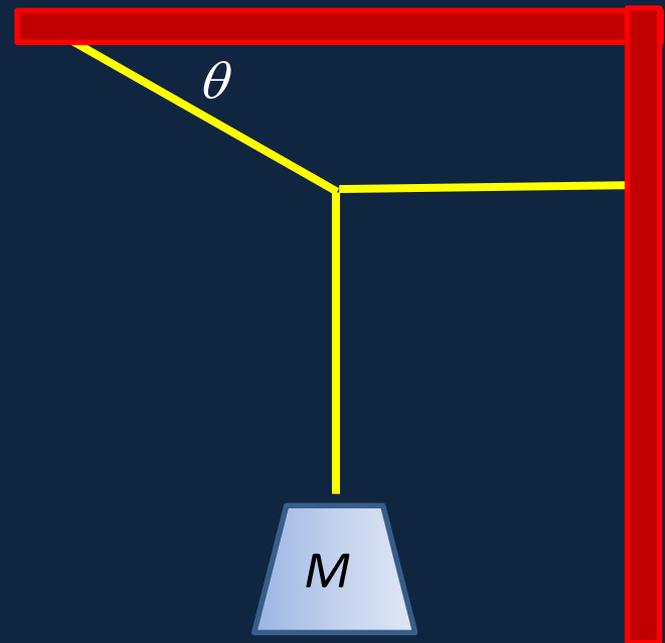
• What is the tension  $T$  in the horizontal string?

A.  $Mg\cos\theta$

B.  $Mg\tan\theta$

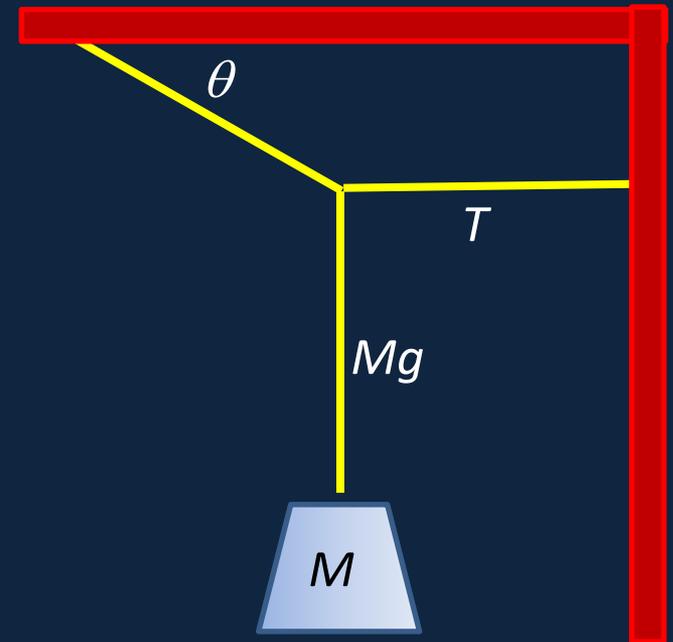
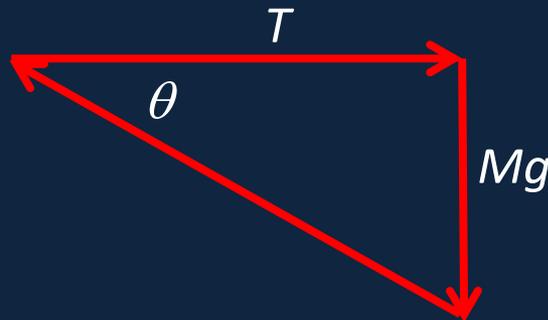
C.  $Mg\cot\theta$

D. None of the above.



# Clicker Answer

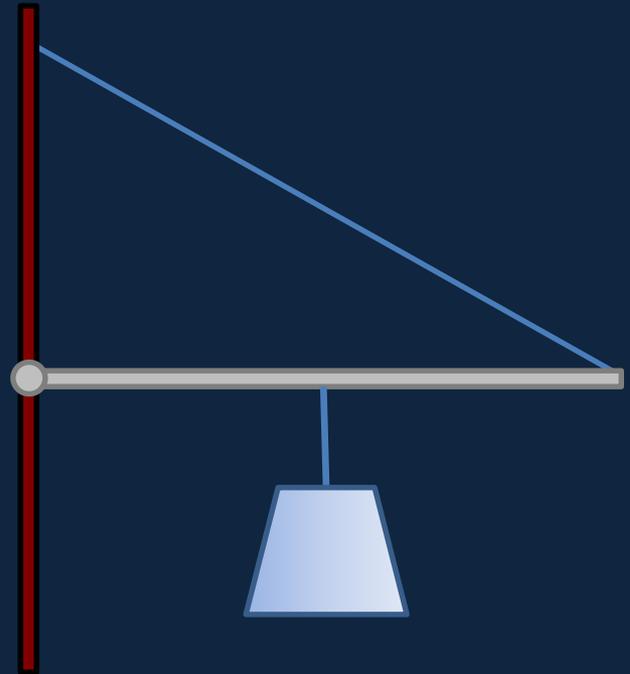
Free body diagram for the **knot** where the three strings meet: the vector tension forces on it must add to zero.



Evidently,  $Mg/T = \tan \theta$ ,  $T = Mg \cot \theta$ .

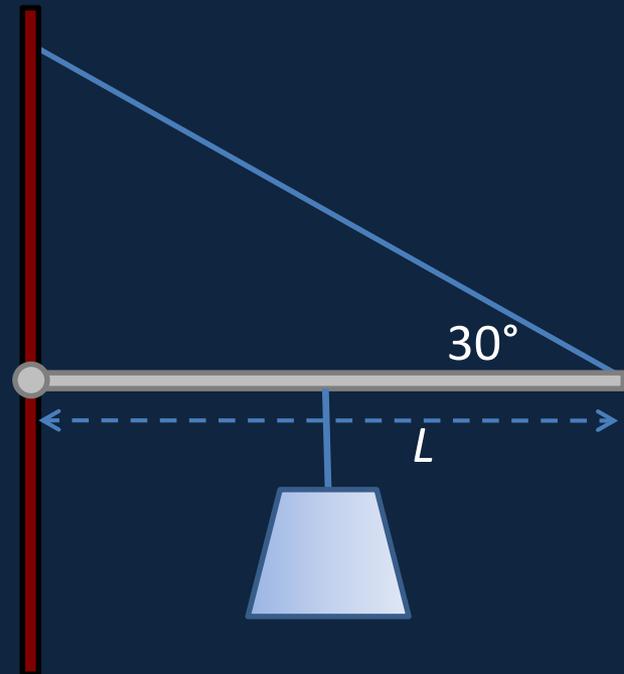
# Clicker Question

- What is the approx tension  $T$  in the top string, given the mass is 2 kg, and it's hung from the midpoint of the rod, which is light and hinged, the angle is  $30^\circ$ ?
- A. 10 N
- B. 20 N
- C.  $20\sqrt{3}$  N
- D. 40 N



# Clicker Answer

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The distance from the hinge to the line of action of the force is  $L\sin 30 = L/2$ .  
Alternatively, the component of the tension force perpendicular to the rod is  $T\sin 30 = T/2$ .