Gauss' Law and Applications

Physics 2415 Lecture 5

Michael Fowler, UVa

Today’s Topics

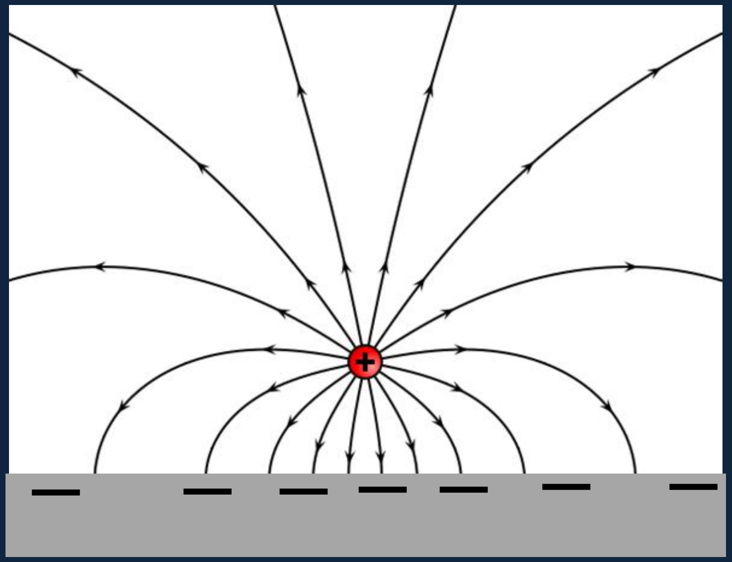
- Gauss’ Law: where it came from—review
- Gauss’ Law for Systems with Spherical Symmetry
- Gauss’ Law for Cylindrical Systems: Coaxial Cable
- Gauss’ Law for Flat Plates

Clicker Question

- A charge +Q is placed a small distance d from a large flat conducting surface.
- Describe the electric field lines: close to the charge, they point radially outwards from the charge, but as they approach the conducting plane:
- A. they bend away from it.
- B. they reach it and just stop.
- C. they curve around to meet the plane at right angles.

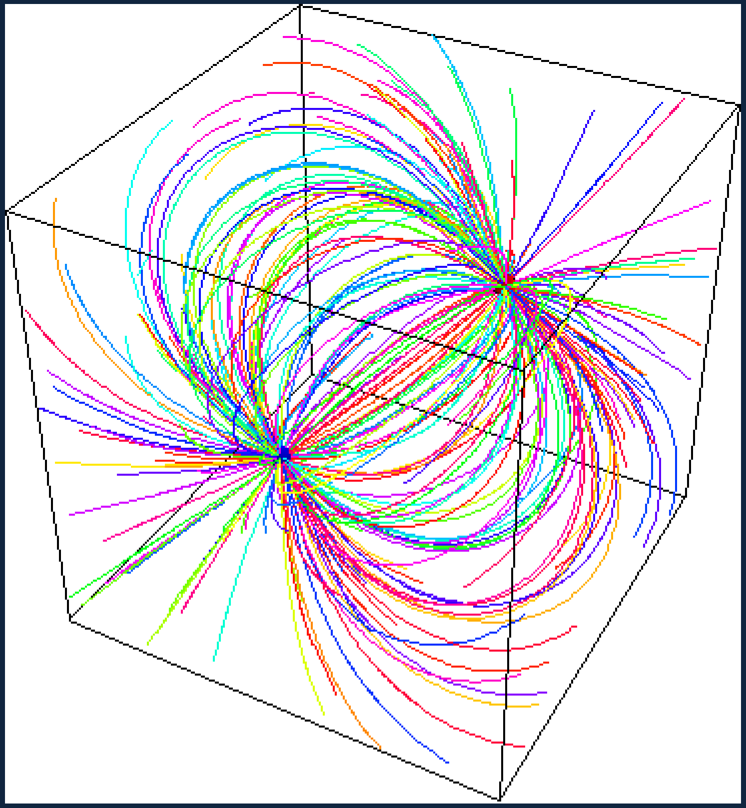
Clicker Answer

- Field lines must always meet a conductor at right angles in electrostatics.
- Physically, the positive charge has attracted negative charges in the conductor to gather in the area under it. They repel each other, so are rather spread out.



Dipole Field Lines in 3D

- There's an analogy with flow of an incompressible fluid: imagine fluid emerging from a source at the positive charge, draining into a sink at the negative charge.
- The electric field lines are like stream lines, showing fluid velocity direction at each point.



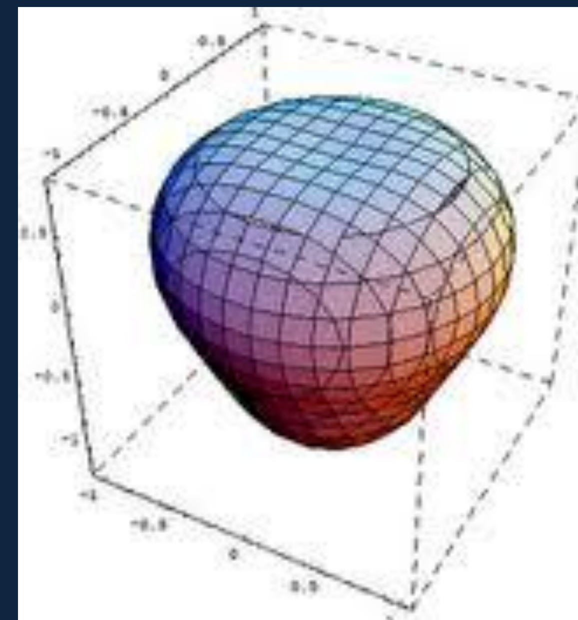
- Check out the applets at http://www.falstad.com/vector2de/ !

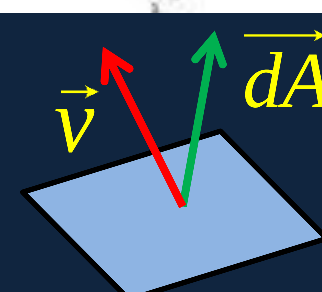
Velocity Field for a Steady Source in 3D

- Imagine you're filling a deep pool, with a hose and its end, deep in the water, is a porous ball so the water flows out equally in all directions.
- Now picture the flow through a spherical fishnet, centered on the source, and far smaller than the pool size.
- Now think of a second spherical net, twice the radius of the first, so 4x the surface area. In steady flow, total water flow across the two spheres is the same: so $v \propto 1/r^2$.
- This velocity field is identical to the electric field from a positive charge!

Total Flow through any Surface

- But how do we *quantify* the fluid flow through such a net?
- We do it one fishnet hole at a time: unlike the sphere, the flow velocity is no longer always perpendicular to the area.
- We represent each fishnet hole by a vector $\overrightarrow{dA}$, magnitude equal to its (small) area, direction perpendicular outwards. Flow through hole is $\vec{v} \cdot \overrightarrow{dA}$
- The total outward flow is $\int_{net} \vec{v} \cdot \overrightarrow{dA}$.





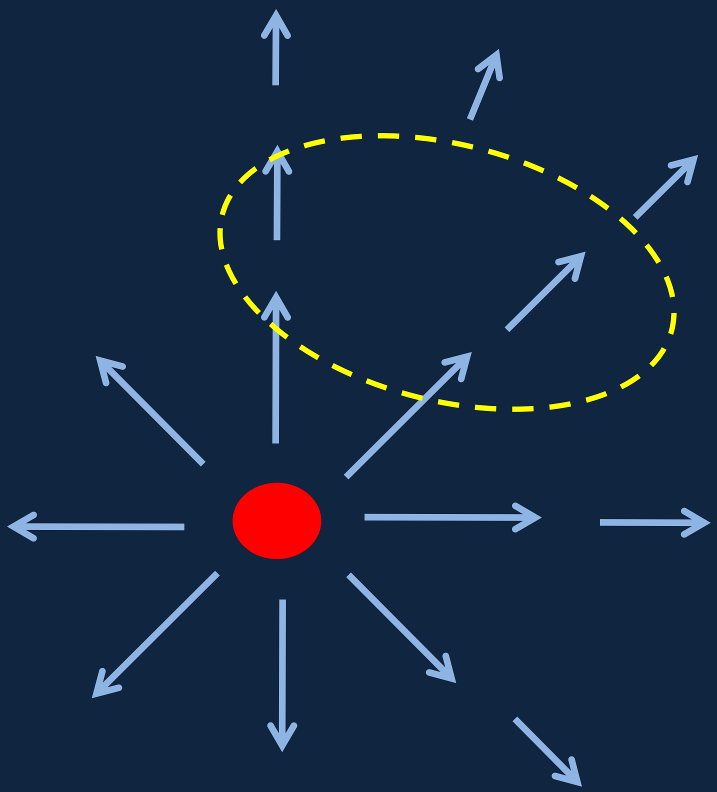
The component of $\vec{v}$ perp. to the surface is $v \cos\theta$.

Gauss's Law

- For incompressible fluid in steady outward flow from a source, the flow rate across any surface enclosing the source $\int \vec{v} \cdot \overrightarrow{dA}$ is the same.
- The electric field from a point charge is identical to this fluid velocity field—it points outward and goes down as $1/r^2$.
- It follows that for the electric field $\vec{E} = \dfrac{1}{4\pi\varepsilon_0}\dfrac{Q\hat{r}}{r^2}$ for any surface enclosing the charge $\int \vec{E} \cdot \overrightarrow{dA} = \text{const.} = Q/\varepsilon_0$ (the value for a sphere).

What about a Closed Surface that *Doesn't* Include the Charge?

- The yellow dotted line represents some fixed closed surface.
- Think of the fluid picture: in steady flow, it goes in one side, out the other. The *net* flow across the surface must be zero—it can't pile up inside.
- By analogy, $\int \vec{E} \cdot \overrightarrow{dA} = 0$ if the charge is outside.



What about More than One Charge?

- Remember the Principle of Superposition: the electric field can always be written as a linear sum of contributions from individual point charges:

$$\vec{E} = \vec{E}_1 + \vec{E}_2 + \vec{E}_3 + \ldots \text{ from } Q_1, Q_2, Q_3 \ldots$$

and so

$$\int \vec{E} \cdot \overrightarrow{dA} = \int \vec{E}_1 \cdot \overrightarrow{dA} + \int \vec{E}_2 \cdot \overrightarrow{dA} + \int \vec{E}_3 \cdot \overrightarrow{dA} + \ldots$$

will have a contribution Q_i / ε_0 from each charge <u>inside</u> the surface—this is Gauss’ Law.

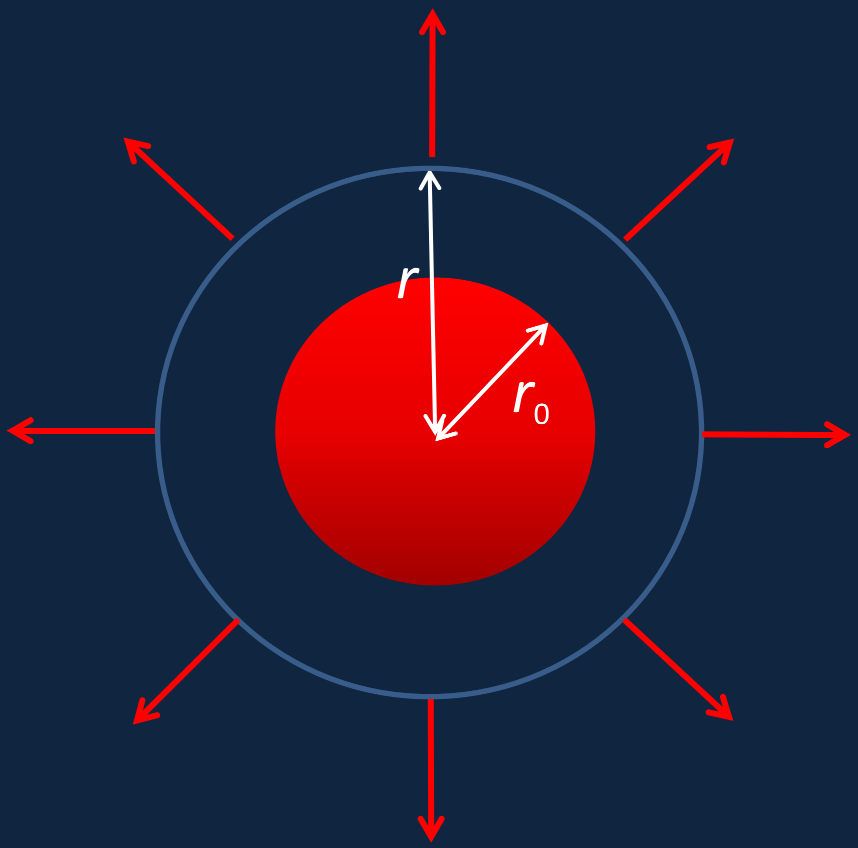
Gauss' Law

- The integral of the total electric field flux out of a closed surface is equal to the total charge Q inside the surface divided by ε_0:

$$\int_S \vec{E} \cdot \overrightarrow{dA} = \frac{Q}{\varepsilon_0}$$

Spherical Symmetry

- First, a uniform spherical shell, radius r_0, of positive charge.
- The perfect spherical symmetry means the electric field outside, at a distance r from the center, must point radially outwards. (rotating the sphere doesn't change anything, but *would* change a field pointing any other way.)

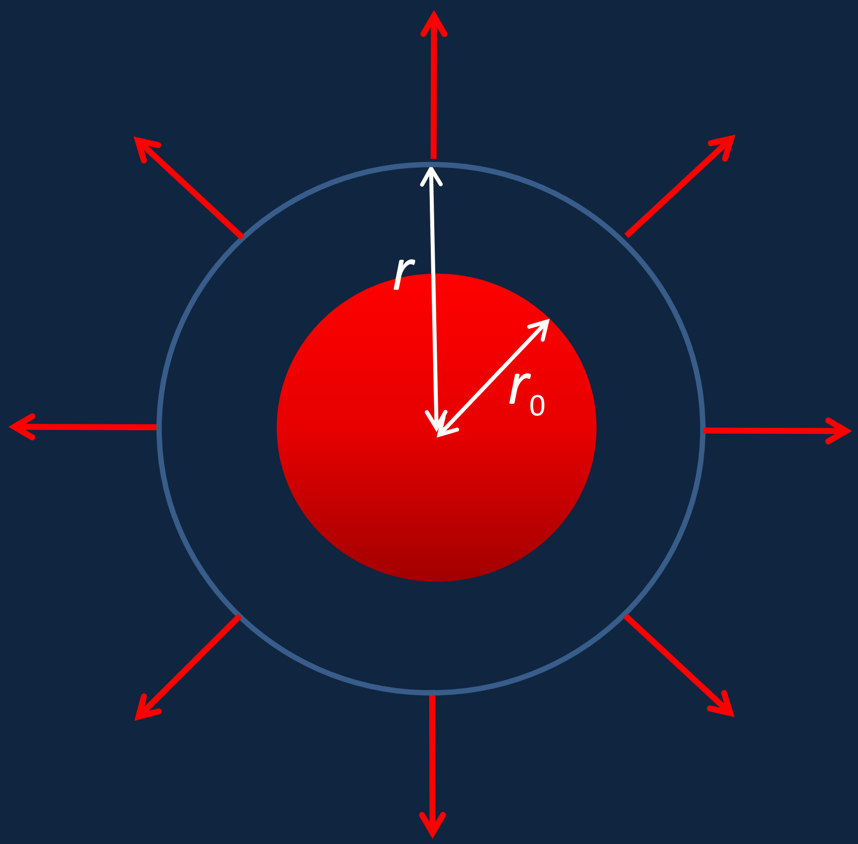


Spherical Symmetry

- The blue circle represents a spherical surface of radius *r*, concentric with the shell of charge.
- For this enclosing surface, Gauss' Law $\int_S \vec{E} \cdot \overrightarrow{dA} = \dfrac{Q}{\varepsilon_0}$ becomes

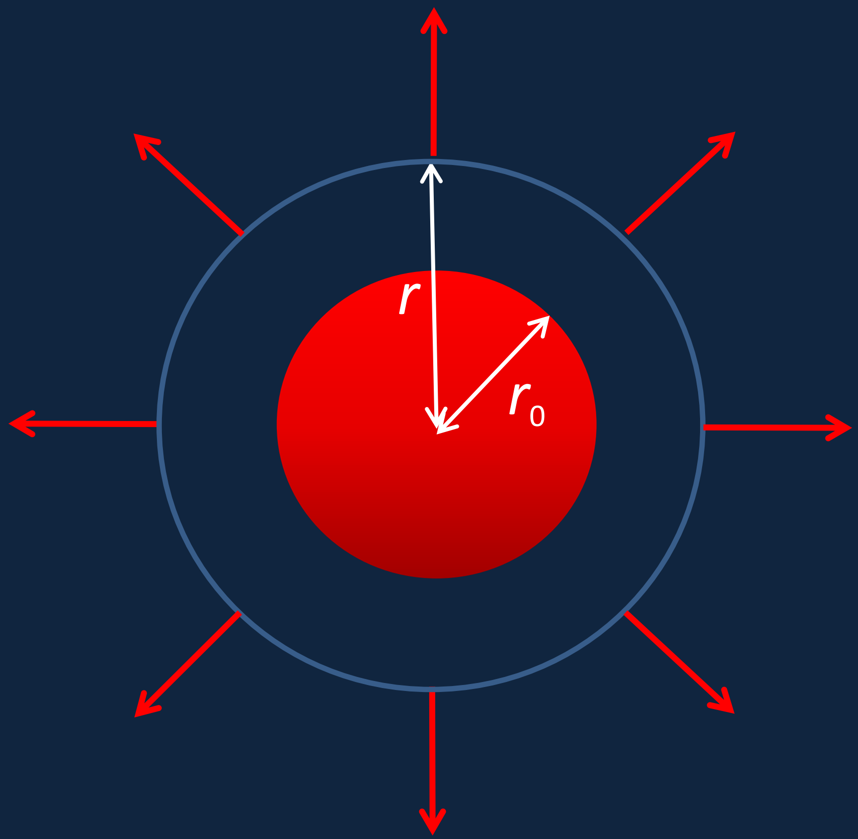
$$4\pi r^2 E(r) = Q / \varepsilon_0,$$

$$E(r) = \frac{1}{4\pi\varepsilon_0}\frac{Q}{r^2} = \frac{kQ}{r^2}$$



Spherical Symmetry

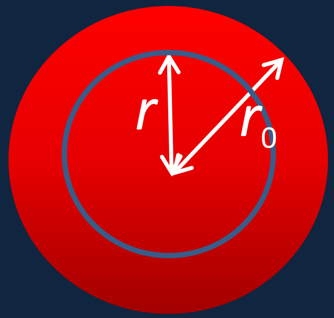
- Gauss' Law easily shows that the electric field from a uniform shell of charge is the same outside the shell as if all the charge were concentrated at a point charge at the center of the sphere. This is difficult to derive using Coulomb's Law!



Field *Inside* a Hollow Shell of Charge

- Now let's take the enclosing surface inside the hollow shell of charge.
- Gauss' Law is now

$$\int_S \vec{E} \cdot \overrightarrow{dA} = \frac{Q}{\varepsilon_0} = 0$$



- Because there is no charge inside the shell, it's all on the surface.
- The spherical symmetry tells us the field inside the shell is exactly zero—again, not so simple from Coulomb's Law.

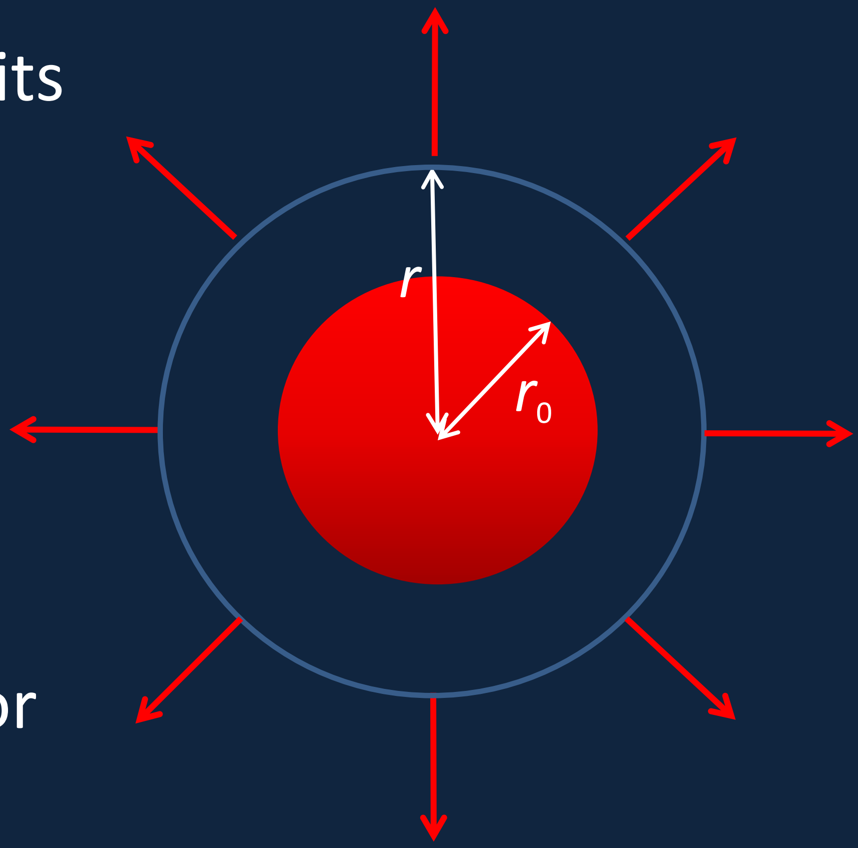
Field Outside a Solid Sphere of Charge

- Assume we have a sphere of insulator with total charge Q distributed uniformly through its volume.
- The field outside is again

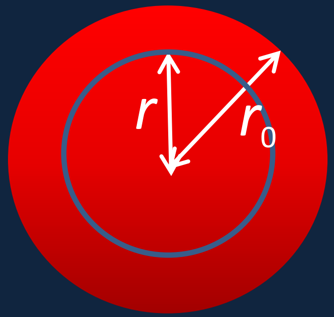
$$E(r) = \frac{1}{4\pi\varepsilon_0}\frac{Q}{r^2}$$

from the spherical symmetry.

- Note: Gauss' Law also works for *gravitation*—and this is the result for a solid sphere of mass.



Field *Inside* a Solid Sphere of Charge



- Now let's take the enclosing surface inside the solid sphere of charge.
- Gauss' Law is now

$$\int_S \vec{E}\cdot\overrightarrow{dA} = \frac{Q_{\text{inside}}}{\varepsilon_0} = Q\times\frac{\text{volume inside } S}{\text{total volume}} = \frac{Q}{\varepsilon_0}\frac{r^3}{r_0^3}$$

- From this, since $\int_S \vec{E}\cdot\overrightarrow{dA} = 4\pi r^2 E$,

$$E = \frac{Q}{4\pi\varepsilon_0}\cdot\frac{r}{r_0^3}$$

so the electric field strength increases linearly from zero at the center to the outside value at the surface.

Clicker Question

How will g change (if at all) on going from the Earth's surface to the bottom of a deep mine? (Assume the Earth has uniform density.)

A. g will be a bit stronger at the bottom of the mine
B. It will be weaker
C. It will be the same as at the surface

Clicker Answer

How will g change (if at all) on going from the Earth's surface to the bottom of a deep mine?

For uniform density, it will be weaker: the gravitational field strength varies in exactly the same way as the electric field from a solid sphere with charge uniformly distributed throughout the volume.

Note: actually the density increases with depth, so things are more complicated…

Clicker Question

- If you could distribute charge perfectly uniformly throughout the volume of a solid spherical conductor, would it stay in place?

A. Yes

B. No

Clicker Answer

- If you could distribute charge perfectly uniformly throughout the volume of a solid spherical conductor, would it stay that way?

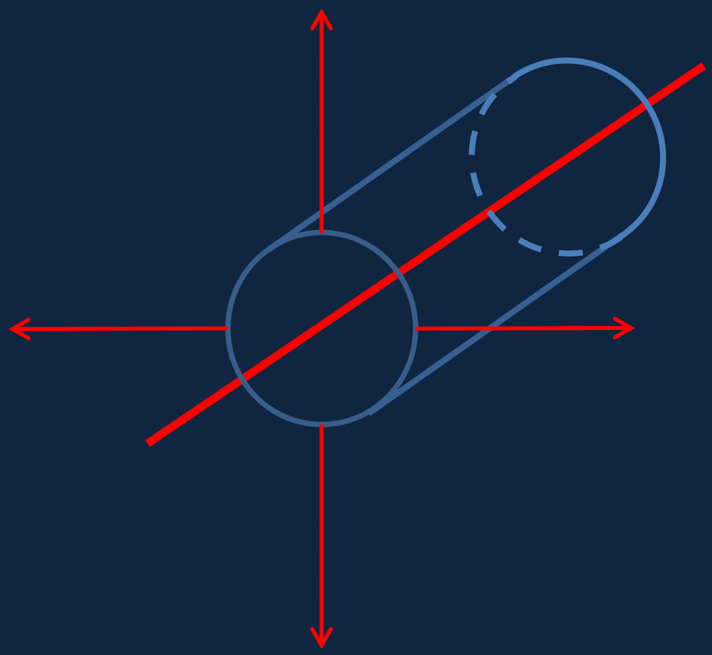
A. Yes

B. No ⟵

Because this charge distribution gives rise to a nonzero outward field *inside* the conductor—the charge would therefore flow radially outwards to the surface.

Field from a Line of Charge

- The field is radially outward from the line, which has charge density λ coul/m.
- Take as gaussian surface a cylinder, radius *r*, axis on the line:
- The flat ends make zero contribution to the surface integral: the electric field vectors lie in the plane.

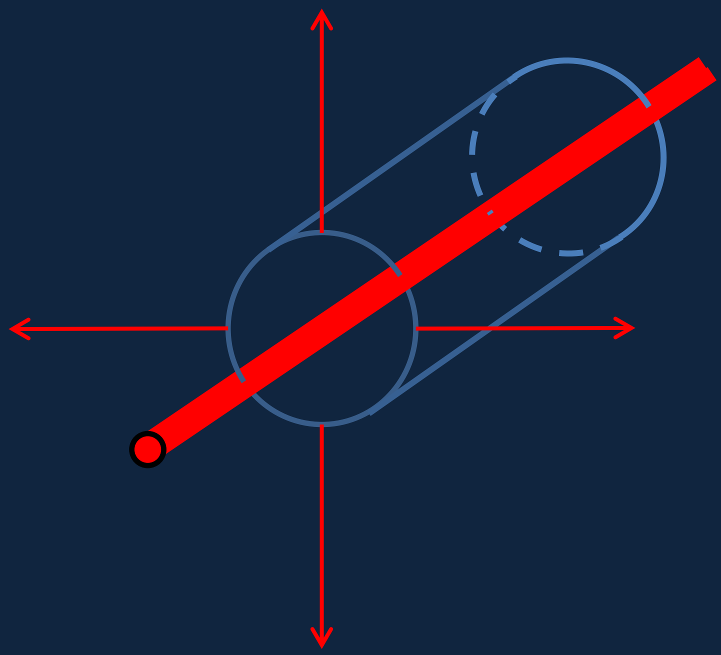


- For the curved surface:

$$\int_S \vec{E}\cdot\overrightarrow{dA} = \frac{Q}{\varepsilon_0} = \frac{\lambda\ell}{\varepsilon_0} = 2\pi r\ell E, \quad E = \frac{\lambda}{\varepsilon_0}\frac{1}{2\pi r} = \frac{2k\lambda}{r}.$$

Field from a Cylinder of Charge

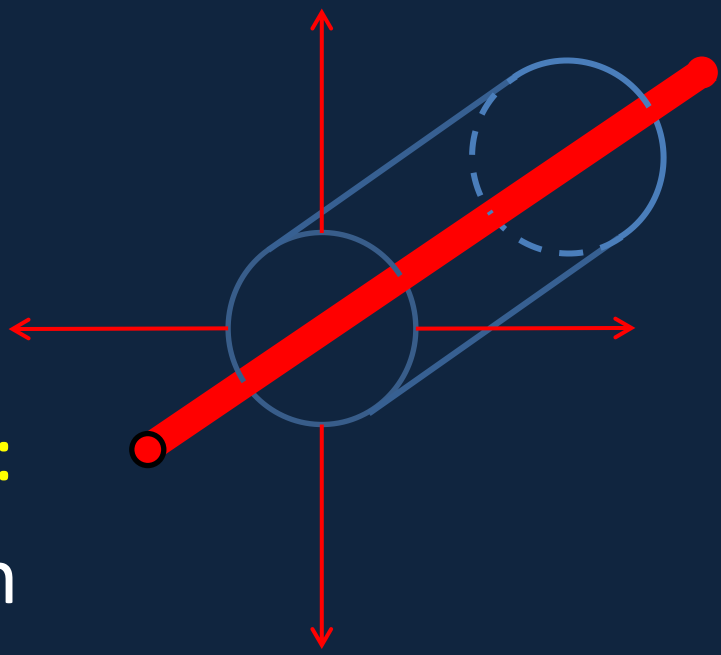
- Taking a gaussian surface as shown, $E = 2k\lambda / r$, exactly as for a line of charge along the center.



Clicker Question

- Suppose the central cylinder is a solid copper rod, carrying charge but with no currents anywhere.
- The charge distribution will be:

A. Uniformly distributed through the rod
B. Restricted to the rod's surface
C. Some other distribution.

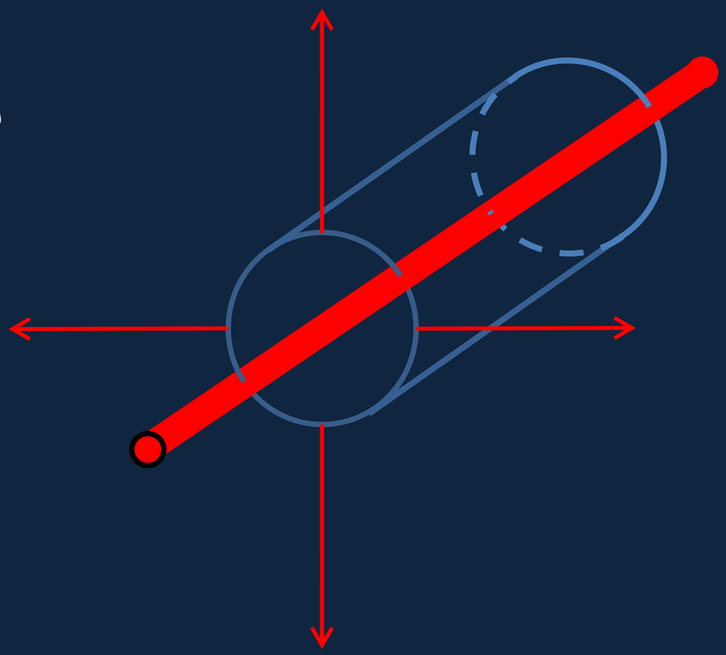


Clicker Answer

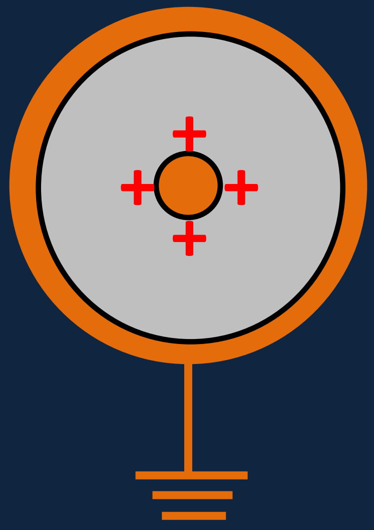
- Suppose the central cylinder is a solid copper rod, carrying charge but with no currents anywhere.
- The charge distribution will be:

Restricted to the rod's surface!

Just like the solid sphere, any charge inside the rod will give rise to an electric field, and therefore a current, flowing outwards.



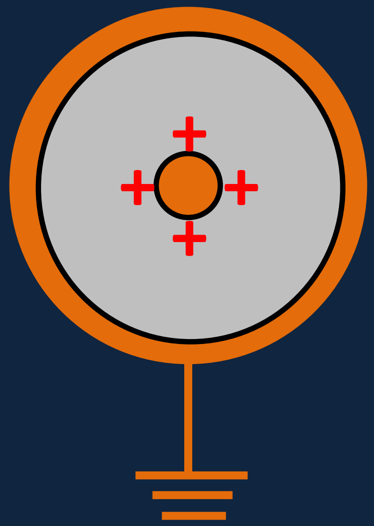
Coaxial Cable Question



- In a coaxial cable, a central conduction cylinder is surrounded by a cylinder of insulator, and *that* is inside a hollow conducting cylinder, which is grounded here.
- If the central conductor is positively charged, the outer conducting cylinder will:

A. have negative charge throughout its volume
B. Have negative charge on its *outside* surface
C. Have negative charge on its *inside* surface
D. Have no net charge.

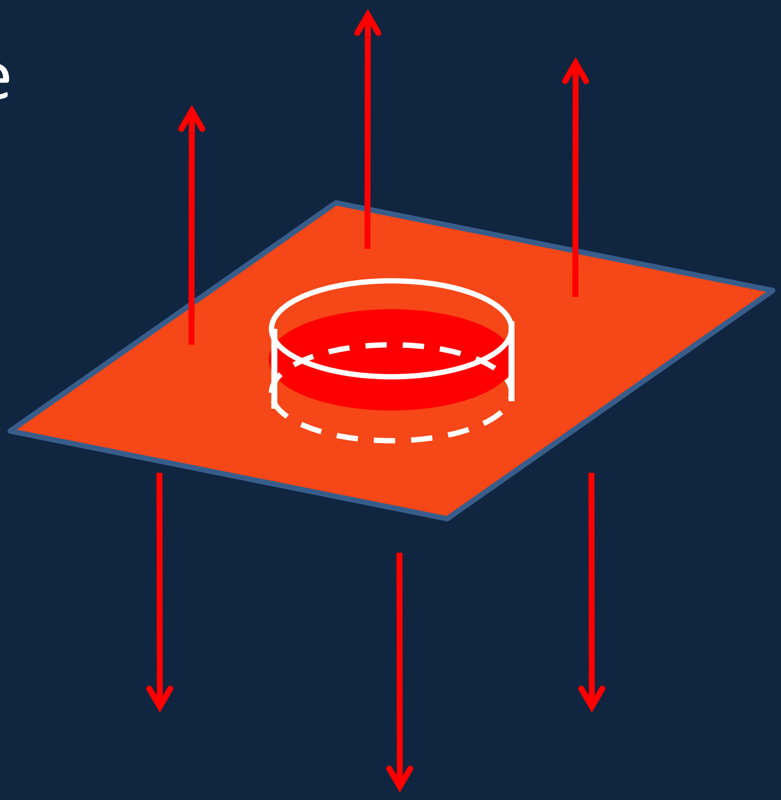
Coaxial Cable Answer



- In a coaxial cable, a central conduction cylinder is surrounded by a cylinder of insulator, and *that* is inside a hollow conducting cylinder, which is grounded here.
- If the central conductor is positively charged, the outer conducting cylinder will:
- Have negative charge on its *inside* surface
- The electric field lines radiating out from the inner conductor must end at the inner surface—there can be no field inside the metal of the outer cylinder.

Uniform Sheet of Charge

- We know from symmetry that the electric field is perpendicularly outward from the plane.
- We take as gaussian surface a "pillbox": shaped like a penny, its round faces parallel to the surface, one above and one below, area *A*. It contains charge (shaded red) $Q = \sigma A$ where the charge density is σ C/m^2.
- Gauss' theorem gives $2AE = \frac{\sigma A}{\varepsilon_0}$, so $E = \frac{\sigma}{2\varepsilon_0}$



Both faces contribute

Charge on Surface of a Conductor

- For a flat conducting surface, the electric field is perpendicularly outward, or a current would arise.
- We have a sheet of charge on the surface, so we take the same Gaussian pillbox as for the sheet of charge, but this time there is no electric field pointing downwards into the conductor.
- Therefore Gauss' Law gives

$$AE = \frac{\sigma A}{\varepsilon_0}, \text{ so } \boxed{E = \frac{\sigma}{\varepsilon_0}}$$