

# AC Circuits II

Physics 2415 Lecture 23

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# Today's Topics

- Review self and mutual induction
- *LR* Circuits
- *LC* Circuits

# Definition of Self-Inductance

- For any shape conductor, when the current changes there is an induced emf  $\mathcal{E}$  opposing the change, and  $\mathcal{E}$  is proportional to the rate of change of current.
- The self-inductance  $L$  is **defined** by:

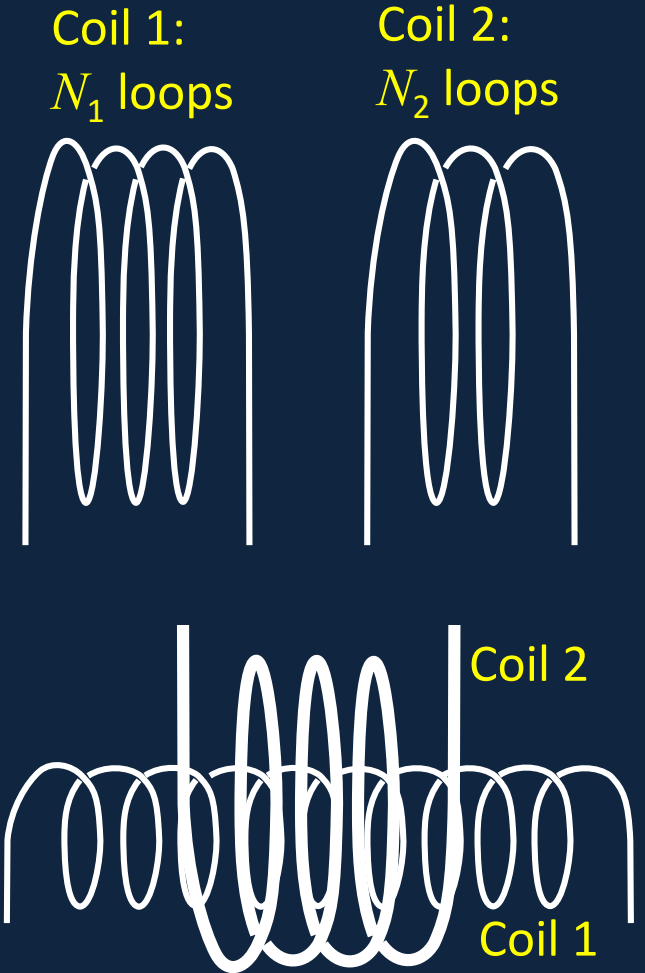
$$\mathcal{E} = -L \frac{dI}{dt}$$

- and symbolized by: 
- Unit: for  $\mathcal{E}$  in volts,  $I$  in amps  $L$  is in henrys (H).

# Mutual Inductance

- We've already met mutual inductance: when the current  $I_1$  in **coil 1** changes, it gives rise to an emf  $\mathcal{E}_2$  in **coil 2**.
- The mutual inductance  $M_{21}$  is defined by:  $M_{21} = N_2 \Phi_{21} / I_1$  where  $\Phi_{21}$  is the magnetic flux through a **single loop** of **coil 2** from current  $I_1$  in **coil 1**.

$$\mathcal{E}_2 = -N_2 \frac{d\Phi_{21}}{dt} = -M_{21} \frac{dI_1}{dt}$$



# Mutual Inductance Symmetry

- Suppose we have two coils close to each other. A changing current in coil 1 gives an emf in coil 2:

$$\mathcal{E}_2 = -M_{21} dI_1 / dt$$

- Evidently we will also find:

$$\mathcal{E}_1 = -M_{12} dI_2 / dt$$

- Remarkably, it turns out that

$$M_{12} = M_{21}$$

- This is by no means obvious, and in fact quite difficult to prove.

# Mutual Inductance and Self Inductance

- For a system of two coils, such as a transformer, the mutual inductance is written as  $M$ .
- Remember that for such a system, emf in one coil will be generated by changing currents in **both** coils:

$$\mathcal{E}_1 = -L_1 \frac{dI_1}{dt} - M \frac{dI_2}{dt}$$

$$\mathcal{E}_2 = -M \frac{dI_1}{dt} - L_2 \frac{dI_2}{dt}$$

# Energy Stored in an Inductance

- If an increasing current  $I$  is flowing through an inductance  $L$ , the emf  $LdI/dt$  is opposing the current, so the source supplying the current is doing work at a rate  $ILdI/dt$ , so to raise the current from zero to  $I$  takes total work

$$U = \int_0^I LI dI = \frac{1}{2} LI^2$$

- This energy is stored in the inductor exactly as  $U = \frac{1}{2} CV^2$  is stored in a capacitor.

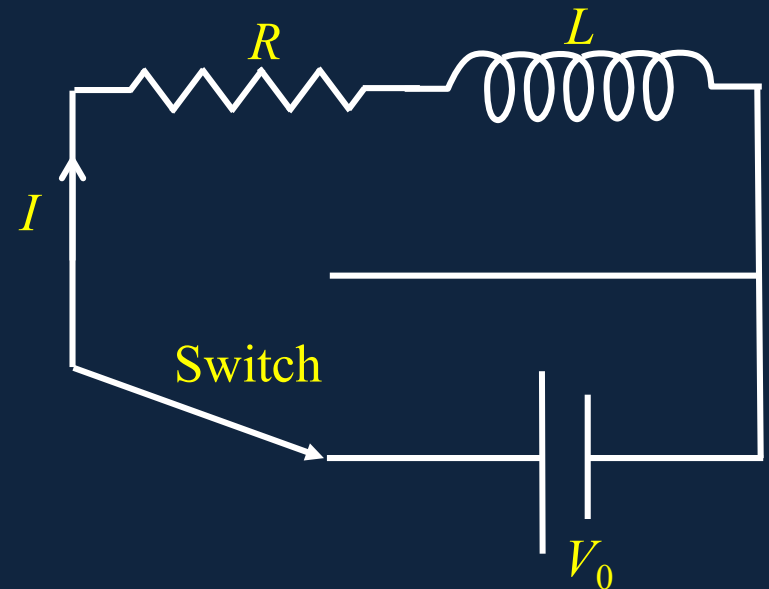
# Energy is Stored in Fields

- When a capacitor is charged, an electric field is created.
- The capacitor's energy is stored in the field with energy density  $\frac{1}{2} \epsilon_0 E^2$ .
- When a current flows through an inductor, a magnetic field is created.
- The inductor's energy is stored in the field with energy density  $\frac{1}{2} B^2 / \mu_0$ .



# *LR* Circuits

- Suppose we have a steady current flowing from the battery through the *LR* circuit shown.
- Then at  $t = 0$  we flip the switch...
- This just takes the battery out of the circuit.



# LR Circuits

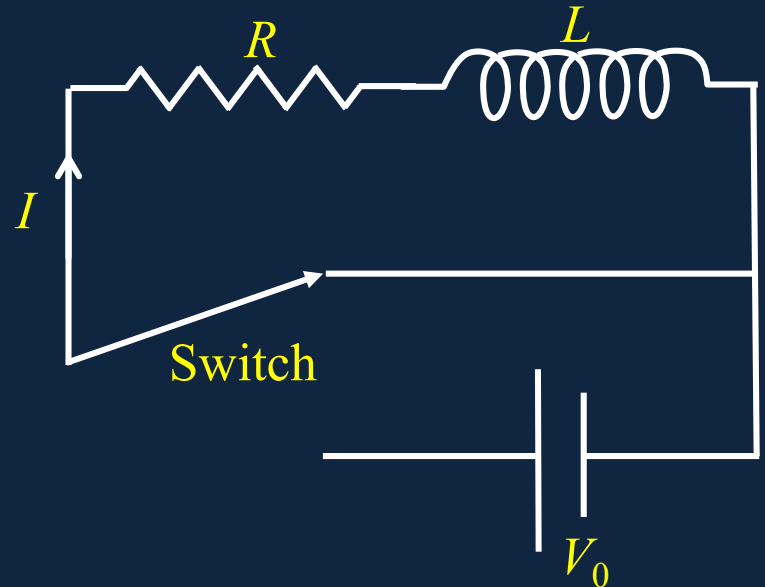
- The decaying current generates an emf

$$\mathcal{E} = -LdI / dt$$

and this drives the current through the resistance:

$$LdI / dt = -IR$$

- This is our old friend  $dx / dt = -ax$  which has solution  $x = x_0 e^{-at}$ .



# LR Circuits

- The equation

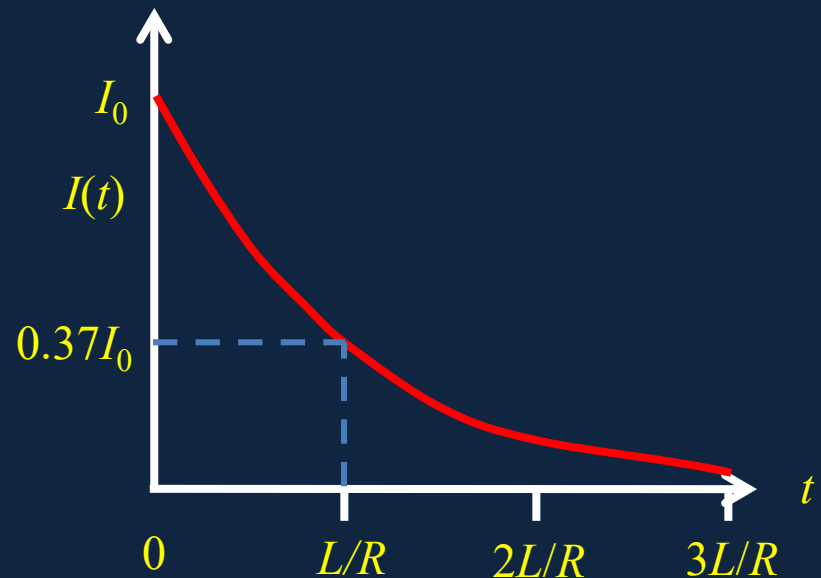
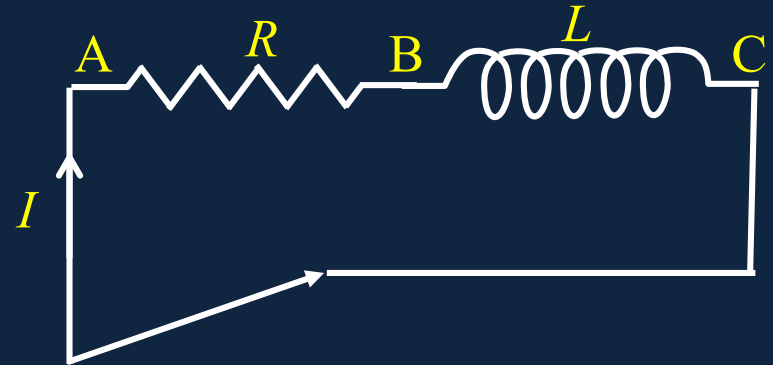
$$L \frac{dI}{dt} = -IR$$

has solution

$$I = I_0 e^{-(R/L)t} = I_0 e^{-t/\tau}$$

so the decay time:

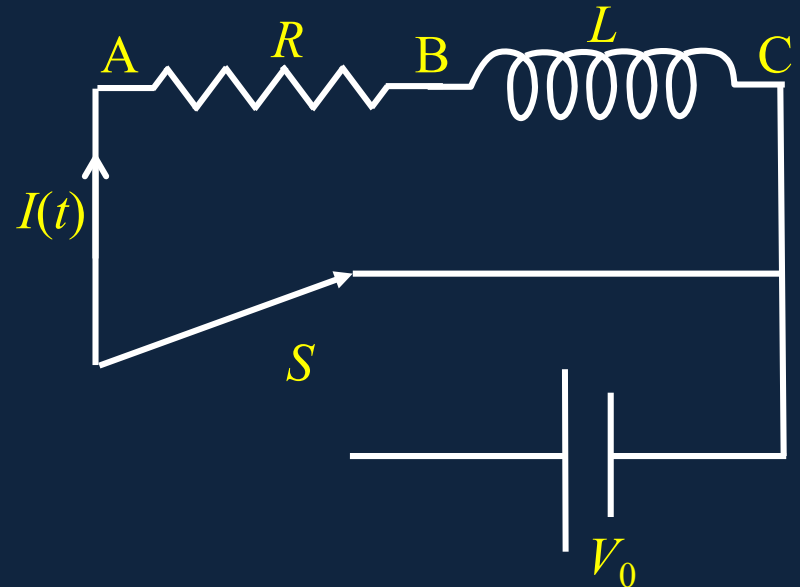
$$\tau = L / R$$



# *LR* Circuits continued...

- Suppose with no initial current we now **reconnect** to the battery.
- **How fast does the current build up?**
- Remember that now the inductance is **opposing** the battery:

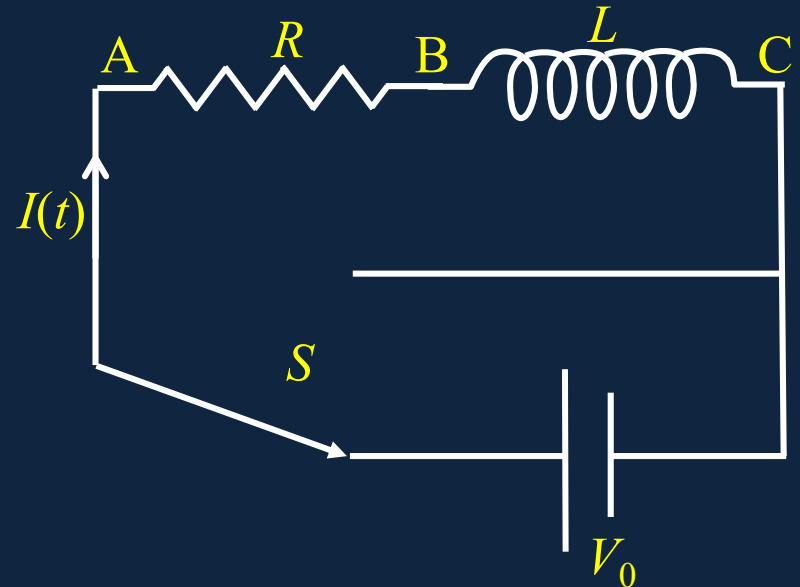
$$V_0 - LdI / dt = IR$$



# *LR* Circuits continued...

- Suppose with no initial current we now **reconnect** to the battery.
- **How fast does the current build up?**
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# LR Circuits continued...

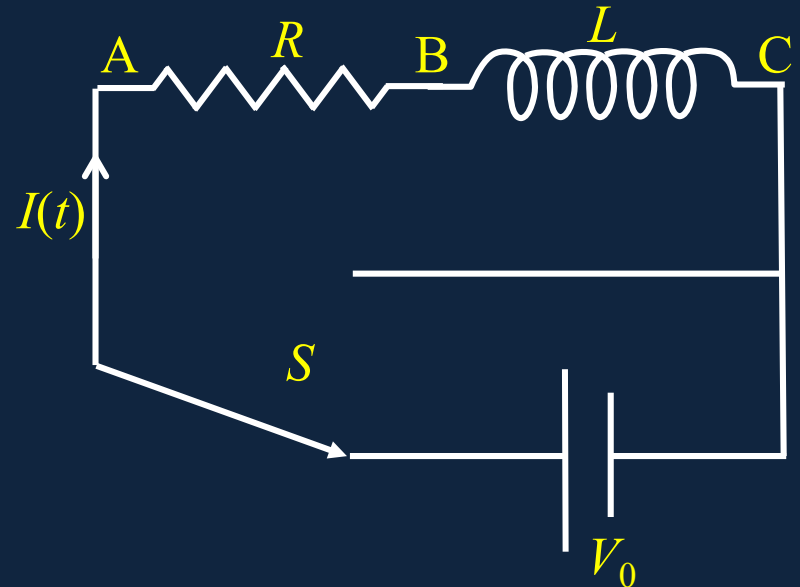
- We must solve the equation

$$V_0 - LdI / dt = IR$$

or

$$dI / dt = -(R / L)I + V_0 / L$$

This differs from the earlier equation by having a constant term added on the right. It's like  $dy / dx = -ay + b$  which you can easily check has solution  $y = Ae^{-ax} + b / a$ .



# *LR* Circuits continued...

- We're solving  $dI / dt = -(R / L)I + V_0 / L$
- We know the solution to  $dy / dx = -ay + b$  is  $y = Ae^{-ax} + b / a$ , where  $A$  is a constant to be fixed by the initial conditions.
- Equating  $I \equiv y$ ,  $t \equiv x$ ,  $R / L \equiv a$ ,  $V_0 / L \equiv b$  gives  $I = Ae^{-(R/L)t} + V_0 / R$  and  $A$  is fixed by the requirement that the current is zero initially, so

$$I = \frac{V_0}{R} \left( 1 - e^{-t/\tau} \right), \quad \tau = L / R$$

# LR Circuits continued...

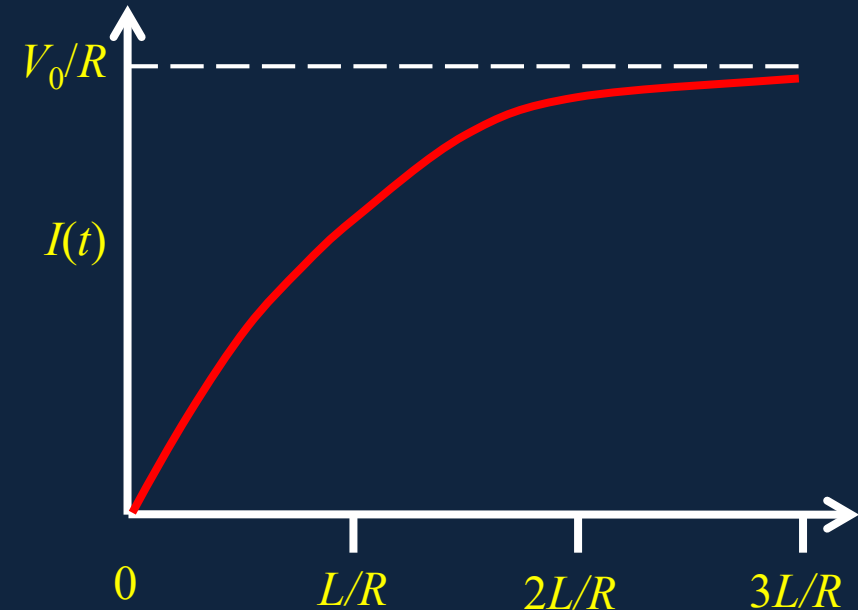
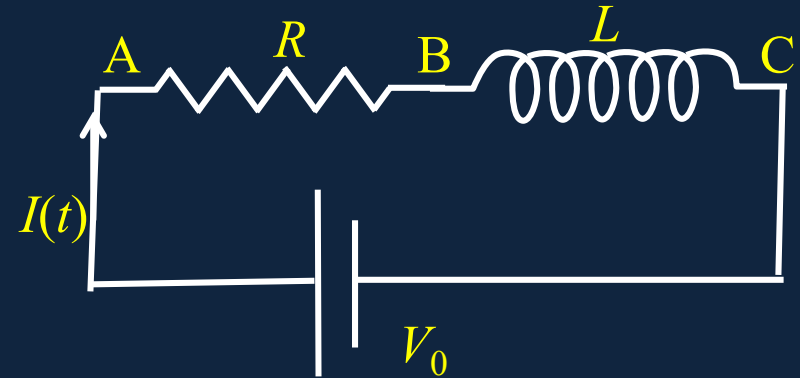
- We've solved

$$LdI / dt = -RI + V_0$$

and found

$$I = \frac{V_0}{R} \left(1 - e^{-t/\tau}\right), \quad \tau = L/R$$

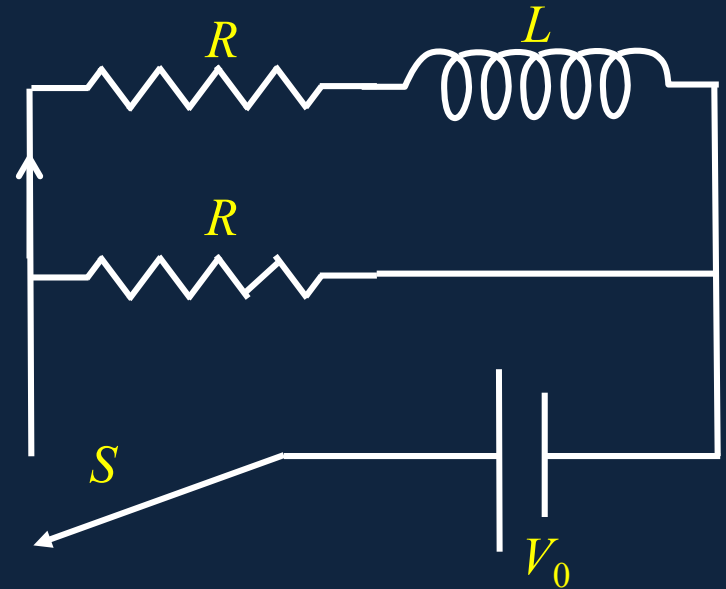
- Initially the current is zero but **changing rapidly**—the inductance emf is equal and opposite to the battery.





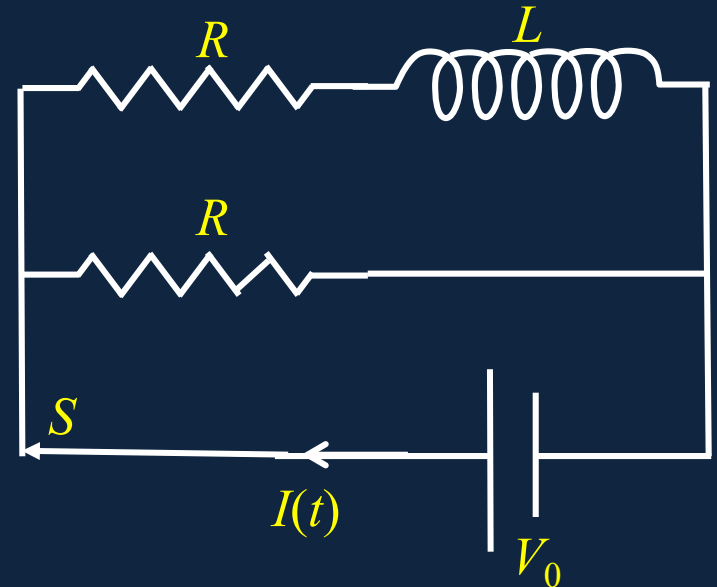
# Clicker Question

- The switch  $S$  is closed...



# Clicker Question

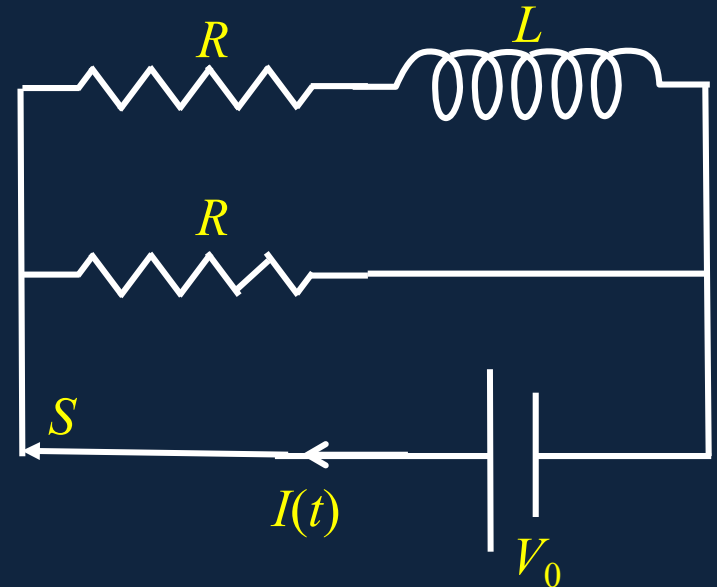
- The switch  $S$  is closed and current flows.
- The initial current, immediately after the switch is closed, is:
  - A  $V_0 / R$
  - B  $2V_0 / R$
  - C  $V_0 / 2R$



# Clicker Answer

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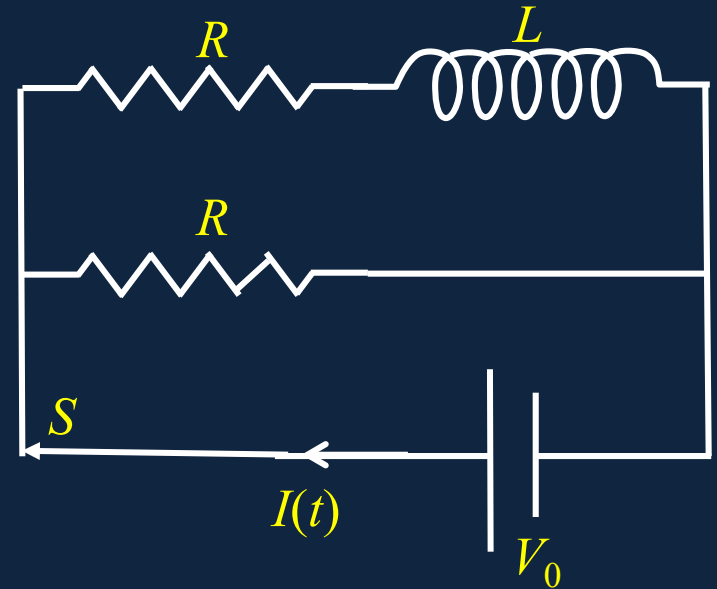


The current through the inductance takes time to build up—it begins at zero. But the current through the other  $R$  starts immediately, so at  $t = 0$  there is current around the lower loop only.

# Clicker Question

- The switch  $S$  is closed and current flows.
- What is the current a long time later?

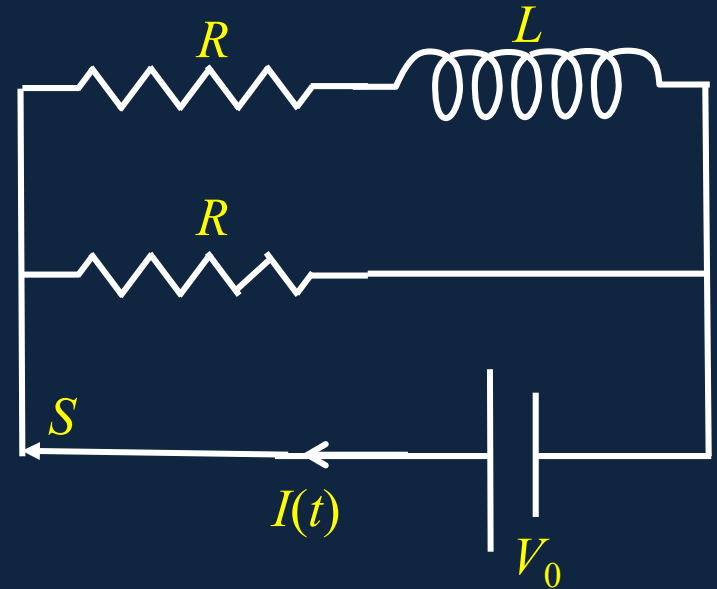
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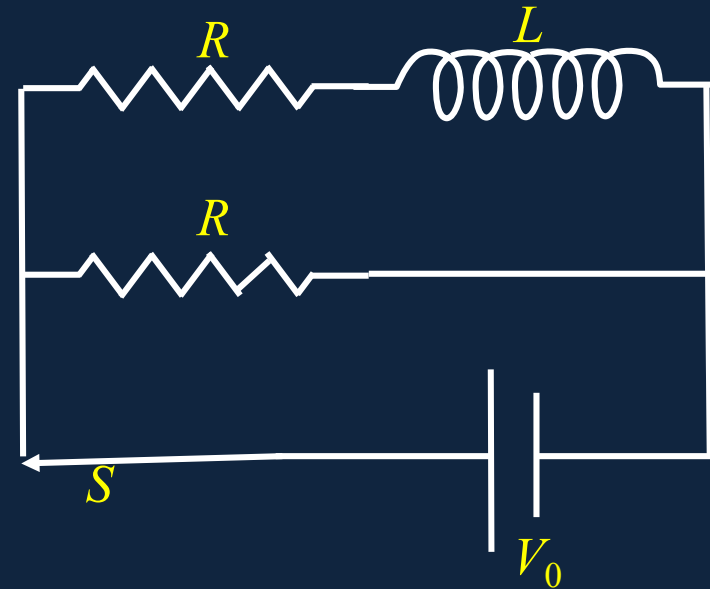
- A  $V_0 / R$
- B  $2V_0 / R$  ←
- C  $V_0 / 2R$



After the current has built up to a steady value, the inductance plays no further role as long as the current remains steady.

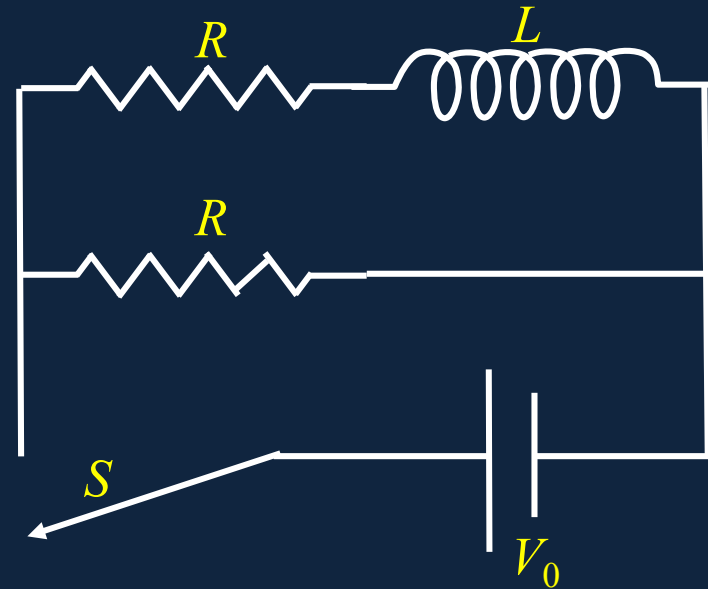
# Clicker Question

- After this long time, the switch is suddenly **opened**!
- What are the currents immediately after the switch is opened?
- A  $V_0 / R$  round the upper loop
- B  $V_0 / 2R$  round the upper loop
- C all currents zero



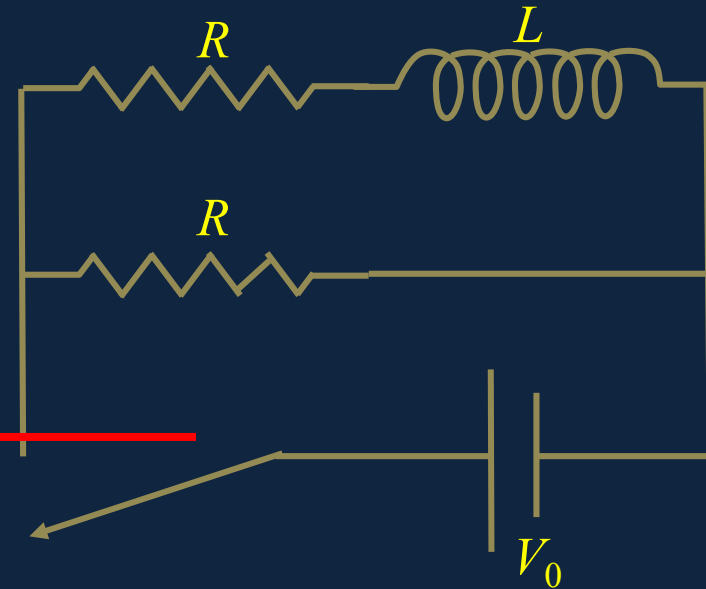
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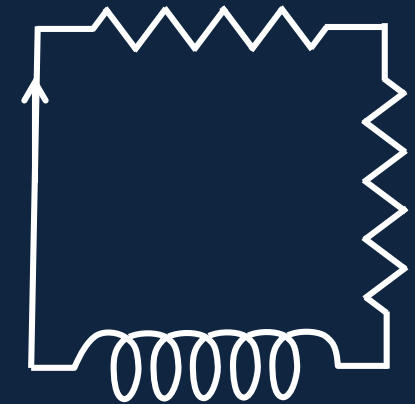
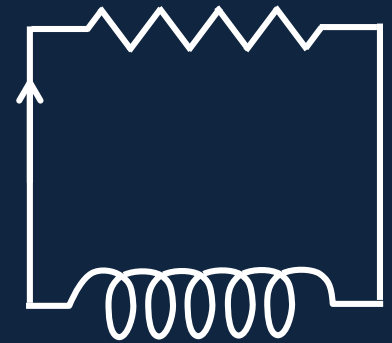


The inductance will not allow sudden discontinuous change in current, so the current through it will be the same just after opening the switch as it was before. This current must now go back via the *other* resistance.



# Clicker Question

- The two circuits shown have the same inductance and the same  $t = 0$  current, no battery, and resistances  $R$  and  $2R$ .
- In which circuit does the current decay more quickly?
  - A.  $R$
  - B.  $2R$
  - C. Both the same



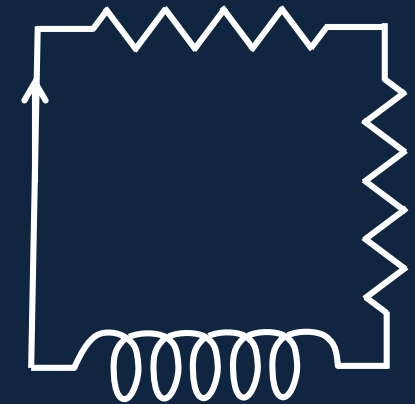
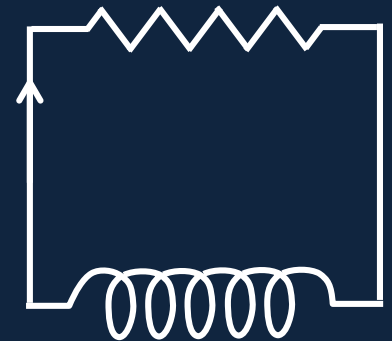
# Clicker Answer

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A.  $R$

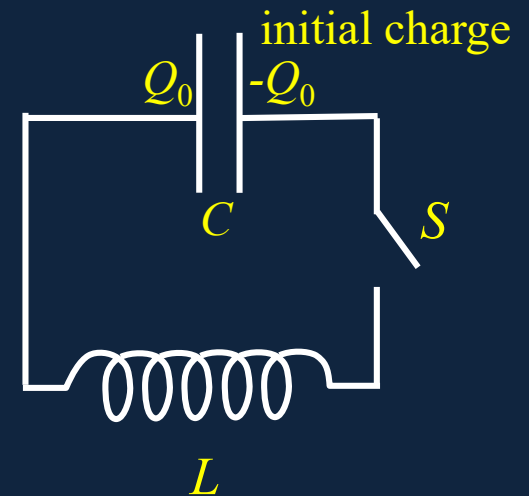
B.  $2R$  ←

The decay is by **heat production**  $I^2R$ .



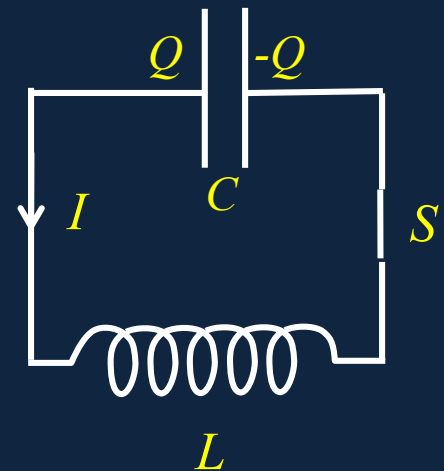
# LC Circuits Question

- Suppose at  $t = 0$  the switch  $S$  is closed, and the resistance in this circuit is extremely small.
- What will happen?
  - A. Current will flow until the capacitor discharges, after which nothing further will happen.
  - B. Current will flow until the capacitor is fully charged the opposite way, then a reverse current will take it back to the original state, etc.



# LC Circuits Answer: B

- This is an **oscillator!**
- The emf  $V = Q/C$  from the capacitor builds up a current through the inductor, so when  $Q$  drops to zero there is substantial current.
- As this current decays, **the inductor generates emf to keep it going**—and with no resistance in the circuit, this is enough to fully charge the oscillator.
- We'll check this out with equations.



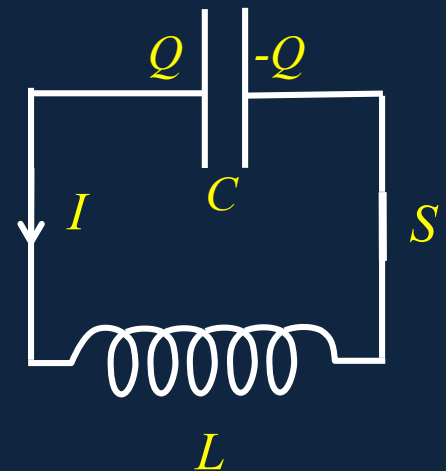
# LC Circuit Analysis

- The current  $I = -dQ / dt$ .
- With no resistance, the voltage across the capacitor is exactly balanced by the emf from the inductance:

$$\frac{Q}{C} = L \frac{dI}{dt}$$

- From the two equations above,

$$\frac{d^2 Q}{dt^2} = -\frac{Q}{LC}$$



$S$  in the diagram is the closed switch

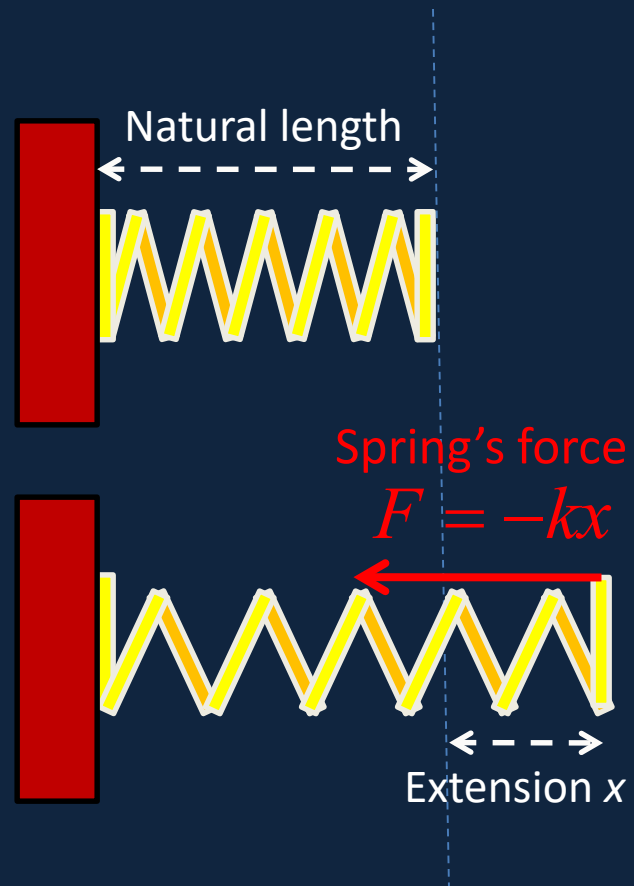
# Force of a Stretched Spring

- If a spring is pulled to extend beyond its natural length by a distance  $x$ , it will pull back with a force

$$F = -kx$$

where  $k$  is called the “spring constant”.

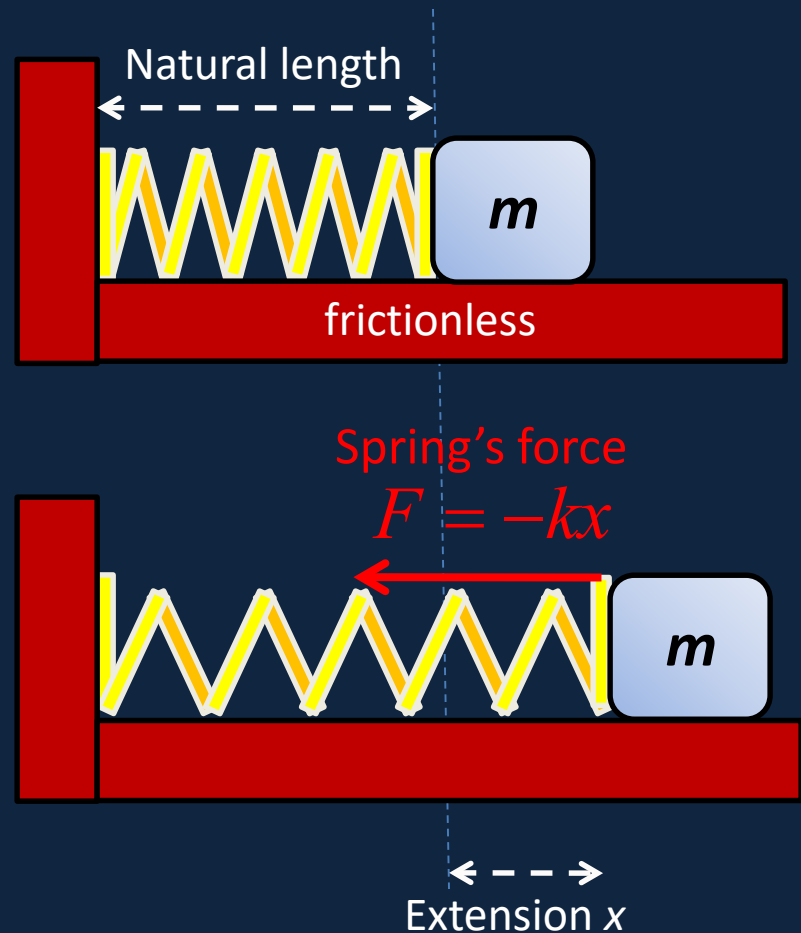
The same linear force is also generated when the spring is *compressed*.



# Mass on a Spring

- Suppose we attach a mass  $m$  to the spring, free to slide backwards and forwards on the frictionless surface, then pull it out to  $x$  and let go.
- $F = ma$  is:

$$m d^2 x / dt^2 = -kx$$



# Solving the Equation of Motion

- For a mass oscillating on the end of a spring,

$$m d^2 x / dt^2 = -kx$$

- The most general solution is

$$x = A \cos(\omega t + \phi)$$

- Here  $A$  is the amplitude,  $\phi$  is the phase, and by putting this  $x$  in the equation,  $m\omega^2 = k$ , or

$$\omega = \sqrt{k / m}$$

- Just as for circular motion, the time for a complete cycle

$$T = 1 / f = 2\pi / \omega = 2\pi \sqrt{m / k} \quad (f \text{ in Hz.})$$



# Back to the $LC$ Circuit...

- The variation of charge with time is

$$\frac{d^2 Q}{dt^2} = -\frac{Q}{LC}$$

- We've just seen that

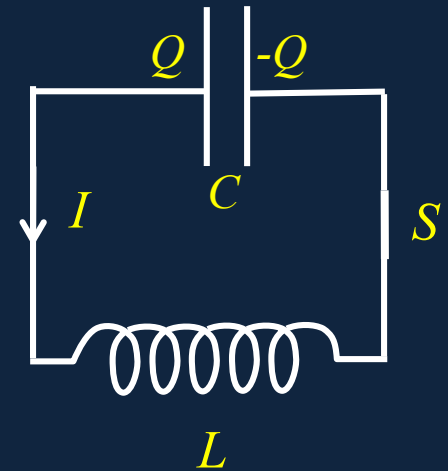
$$m d^2 x / dt^2 = -kx$$

has solution

$$x = A \cos(\omega t + \phi), \quad \omega = \sqrt{k / m}$$

from which

$$Q = Q_0 \cos \omega t, \quad \omega = 1 / \sqrt{LC}.$$



# Where's the Energy in the $LC$ Circuit?

- The variation of charge with time is

$$Q = Q_0 \cos \omega t, \quad \omega = 1 / \sqrt{LC}$$

so the energy stored in the capacitor is

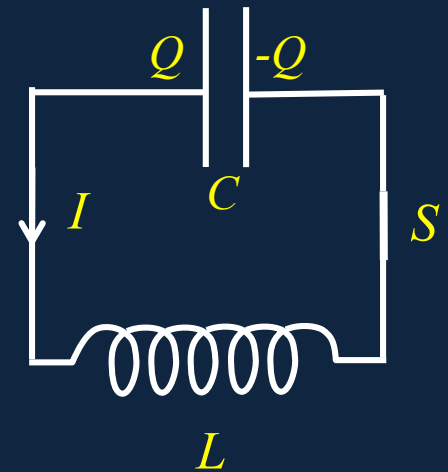
$$U_E = Q^2 / 2C = (Q_0^2 / 2C) \cos^2 \omega t$$

- The current is the charge flowing out

$$I = -dQ / dt = Q_0 \omega \sin \omega t$$

so the energy stored in the inductor is

$$U_B = \frac{1}{2} LI^2 = \frac{1}{2} L Q_0^2 \omega^2 \sin^2 \omega t = (Q_0^2 / 2C) \sin^2 \omega t \quad (\omega^2 = 1 / LC)$$




Compare this with the energy stored in the capacitor!

# Clicker Question

- Suppose an  $LC$  circuit has a very large capacitor but a small inductor (and no resistance).
- During the period of one oscillation, is the maximum energy stored in the inductor
  - A. greater than
  - B. less than
  - C. equal tothe maximum energy stored in the capacitor?

# Clicker Answer

- Suppose an  $LC$  circuit has a very large capacitor but a small inductor (and no resistance).
- During the period of one oscillation, is the maximum energy stored in the inductor
  - A. greater than
  - B. less than
  - C. equal to  the maximum energy stored in the capacitor?

# Energy in the $LC$ Circuit

- We've found the energy in the capacitor is

$$U_E = Q^2 / 2C = (Q_0^2 / 2C) \cos^2 \omega t$$

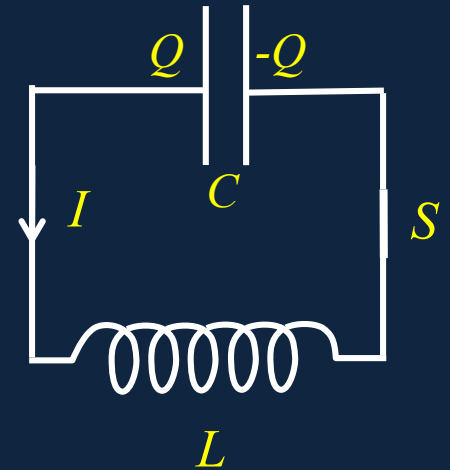
- The energy stored in the inductor is

$$U_B = \frac{1}{2} LI^2 = (Q_0^2 / 2C) \sin^2 \omega t$$

- So the **total energy** is

$$U_B = (Q_0^2 / 2C) (\cos^2 \omega t + \sin^2 \omega t) = Q_0^2 / 2C.$$

- Total energy is of course **constant**: it is cyclically sloshed back and forth between the electric field and the magnetic field.



# Energy in the $LC$ Circuit

- Energy in the capacitor:  
electric field energy
- Energy in the inductor:  
magnetic field energy

