AC Circuits III

Physics 2415 Lecture 24

Michael Fowler, UVa

Today's Topics

- LC circuits: analogy with mass on spring
- LCR circuits: damped oscillations
- LCR circuits with ac source: driven pendulum, resonance.

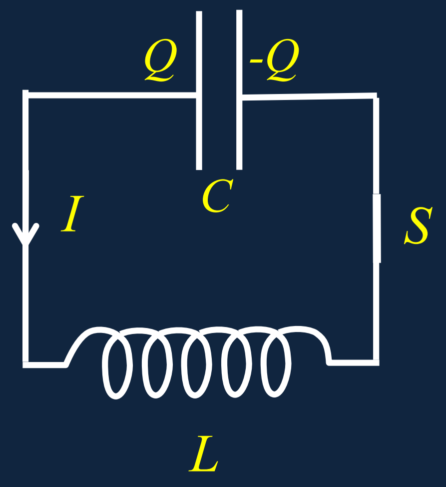
LC Circuit Analysis

- The current $I = -dQ/dt$.
- With no resistance, the voltage across the capacitor is exactly balanced by the emf from the inductance:

$$\frac{Q}{C} = L\frac{dI}{dt}$$

- From the two equations above,

$$\frac{d^2Q}{dt^2} = -\frac{Q}{LC}$$



S in the diagram is the closed switch

Quick review of simple harmonic motion from Physics 1425…

Force of a Stretched Spring

- If a spring is pulled to extend beyond its natural length by a distance *x*, it will pull back with a force

$$F = -kx$$

where *k* is called the "spring constant".

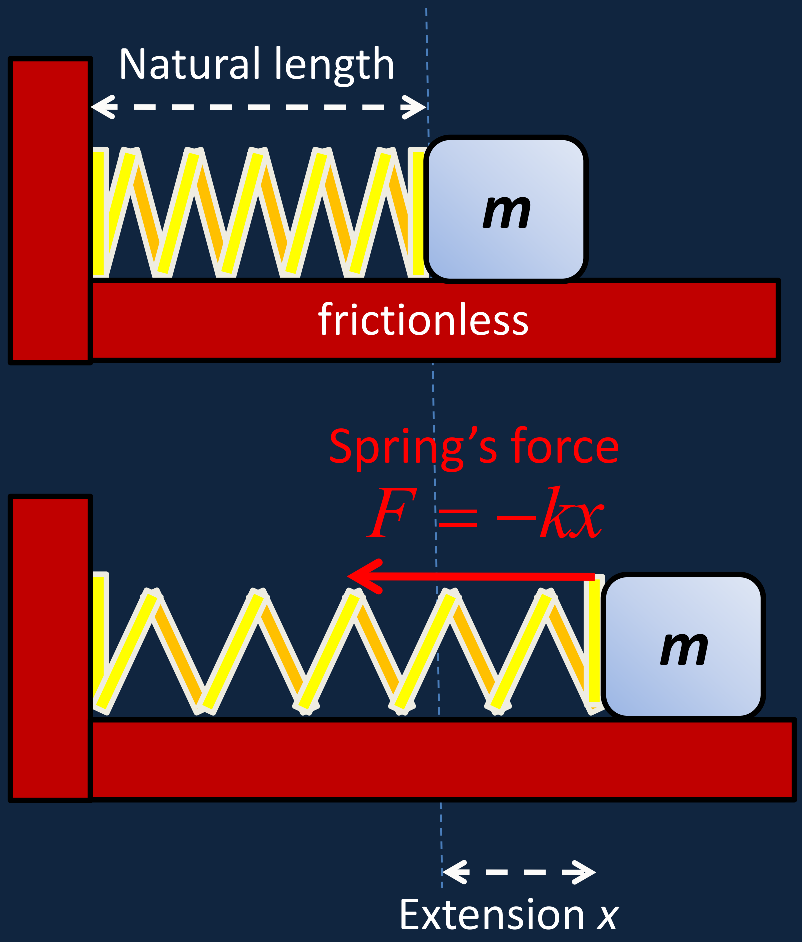
The same linear force is also generated when the spring is *compressed*.

Quick review of simple harmonic motion from Physics 1425…

Mass on a Spring

- Suppose we attach a mass *m* to the spring, free to slide backwards and forwards on the frictionless surface, then pull it out to *x* and let go.
- *F* = *ma* is:

$$md^2x/dt^2 = -kx$$



Quick review of simple harmonic motion from Physics 1425…

Solving the Equation of Motion

- For a mass oscillating on the end of a spring,

$$md^2x/dt^2 = -kx$$

- The most general solution is

$$x = A\cos(\omega t + \phi)$$

- Here *A* is the amplitude, ϕ is the phase, and by putting this *x* in the equation, $m\omega^2 = k$, or

$$\omega = \sqrt{k/m}$$

- Just as for circular motion, the time for a complete cycle

$$T = 1/f = 2\pi/\omega = 2\pi\sqrt{m/k} \quad (f \text{ in Hz.})$$

Back to the LC Circuit…

- The variation of charge with time is

$$\frac{d^2Q}{dt^2} = -\frac{Q}{LC}$$

- We've just seen that

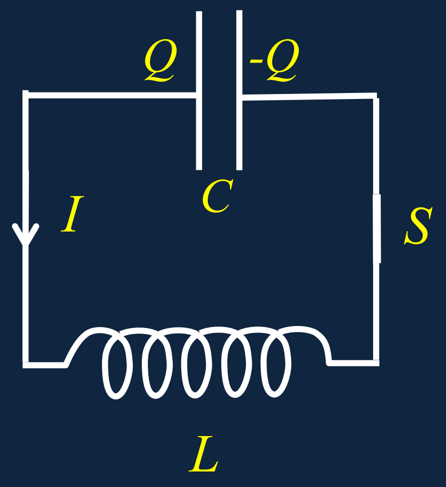
$$md^2x/dt^2 = -kx$$

has solution

$$x = A\cos(\omega t + \phi), \quad \omega = \sqrt{k/m}$$

from which

$$Q = Q_0 \cos \omega t, \quad \omega = 1/\sqrt{LC}.$$



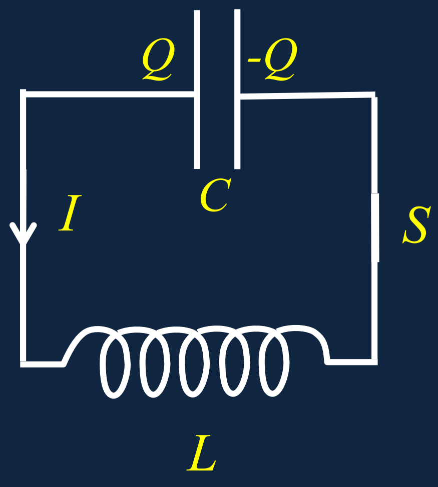
Where's the Energy in the LC Circuit?

- The variation of charge with time is

$$Q = Q_0 \cos \omega t, \quad \omega = 1/\sqrt{LC}$$

so the energy stored in the capacitor is

$$U_E = Q^2/2C = \left(Q_0^2/2C\right)\cos^2 \omega t$$



- The current is the charge flowing out

$$I = -dQ/dt = Q_0 \omega \sin \omega t$$

so the energy stored in the inductor is

$$U_B = \tfrac{1}{2} L I^2 = \tfrac{1}{2} L Q_0^2 \omega^2 \sin^2 \omega t = \left(Q_0^2/2C\right)\sin^2 \omega t \quad \left(\omega^2 = 1/LC\right)$$

Compare this with the energy stored in the capacitor!

Energy in the LC Circuit

- We've found the energy in the capacitor is

$$U_E = Q^2 / 2C = \left(Q_0^2 / 2C\right)\cos^2 \omega t$$

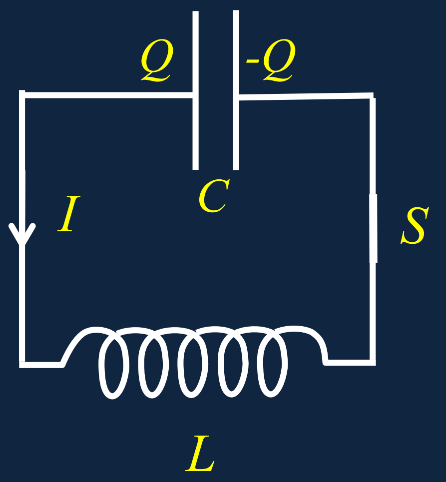
- The energy stored in the inductor is

$$U_B = \tfrac{1}{2} L I^2 = \left(Q_0^2 / 2C\right)\sin^2 \omega t$$

- So the total energy is

$$U_B = \left(Q_0^2 / 2C\right)\left(\cos^2 \omega t + \sin^2 \omega t\right) = Q_0^2 / 2C.$$

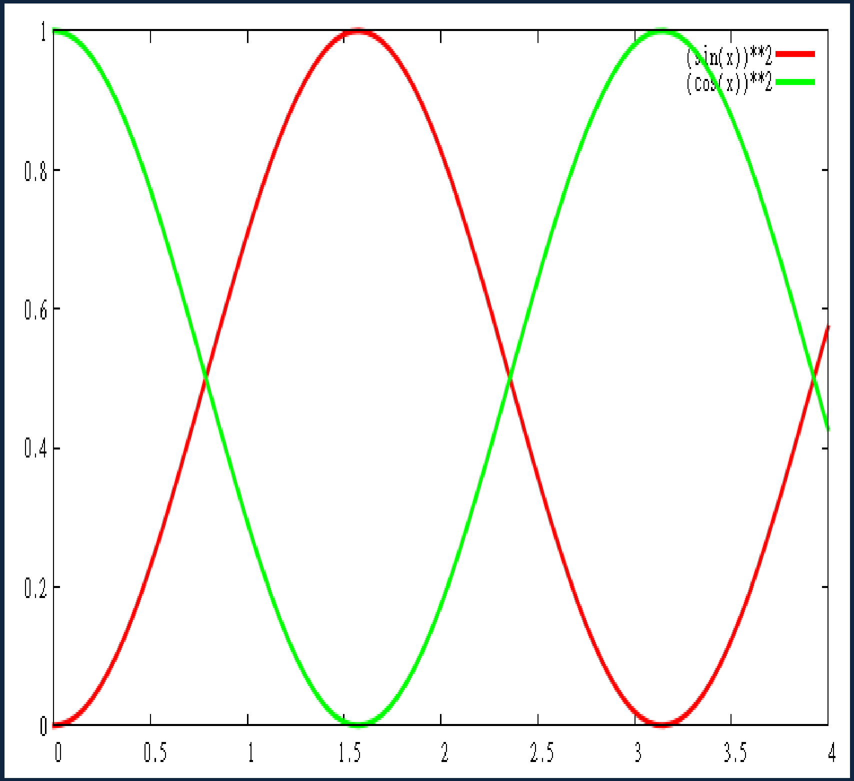
- Total energy is of course constant: it is cyclically sloshed back and forth between the electric field and the magnetic field.



Energy in the LC Circuit

- Energy in the capacitor: electric field energy

- Energy in the inductor: magnetic field energy



The LRC Circuit

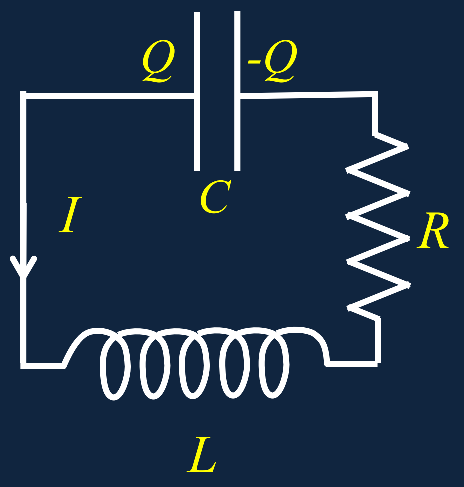
- Adding a resistance R to the LC circuit, adds a voltage drop IR, so

$$\frac{Q}{C} = L\frac{dI}{dt} + IR$$

- Remembering $I = -dQ/dt$, we find

$$L\frac{d^2Q}{dt^2} + R\frac{dQ}{dt} + \frac{Q}{C} = 0.$$

- A differential equation we've seen before…

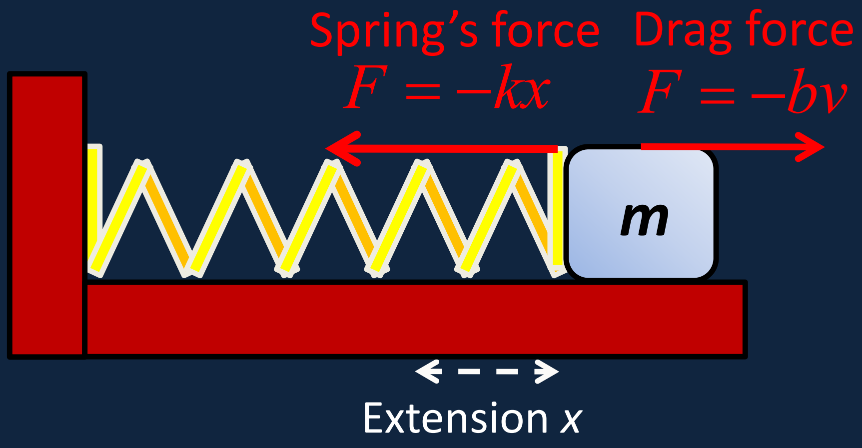


Damped Harmonic Motion

- In the real world, oscillators experience damping forces: friction, air resistance, etc.
- These forces always oppose the motion: as an example, we consider a force $F = -bv$ proportional to velocity.
- Then $F = ma$ becomes:

$$ma = -kx - bv$$

- That is, $md^2x/dt^2 + bdx/dt + kx = 0$



The direction of drag force shown is on the assumption that the mass is moving to the *left*.

LRC is just a Damped Oscillator

- Compare our charge equation with the displacement equation for a damped harmonic oscillator:

$$L\frac{d^2Q}{dt^2}+R\frac{dQ}{dt}+\frac{Q}{C}=0.$$

$$md^2x/dt^2+bdx/dt+kx=0$$

- They are the same:

$$Q\equiv x,\quad L\equiv m,\quad R\equiv b,\quad 1/C\equiv k.$$

Equation Solution

From Physics 1425:

- The equation of motion

$$md^2x/dt^2 + bdx/dt + kx = 0$$

has solution

$$x = Ae^{-\gamma t}\cos\omega' t$$

where

$$\gamma = b/2m,$$

$$\omega' = \sqrt{(k/m) - (b^2/4m^2)}$$

- Therefore

$$L\frac{d^2Q}{dt^2} + R\frac{dQ}{dt} + \frac{Q}{C} = 0$$

has solution

$$Q = Q_0 e^{-\gamma t}\cos\omega' t$$

where

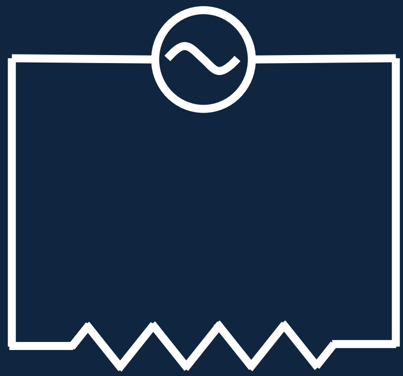
$$\gamma = R/2L,$$

$$\omega' = \sqrt{(1/LC) - (R^2/4L^2)}$$

$$Q \equiv x, \quad L \equiv m, \quad R \equiv b, \quad 1/C \equiv k.$$

Spreadsheet!

AC Source and Resistor

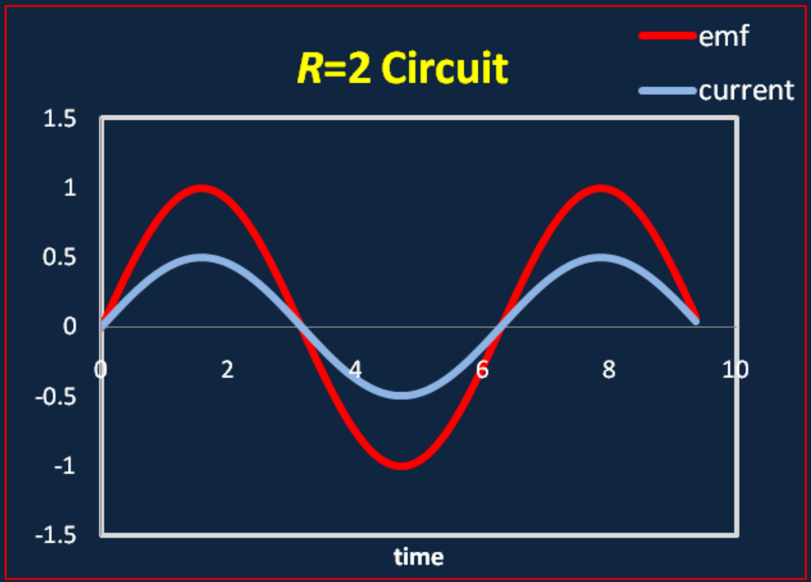


- For an AC source (denoted by a wavy line in a circle) $V = V_0 \sin \omega t$ the current is:

$$I = I_0 \sin \omega t = \left(V_0 / R\right) \sin \omega t.$$

- The current and voltage peak at the same time.
- Power: the ac source is working at a rate

$$\overline{P} = \overline{IV} = I_0 V_0 \overline{\sin^2 \omega t} = \tfrac{1}{2} I_0 V_0$$



AC Source and Inductor

- For a purely inductive circuit, for $V = V_0 \sin \omega t$, the current is given by

$$V_0 \sin \omega t = L dI / dt$$

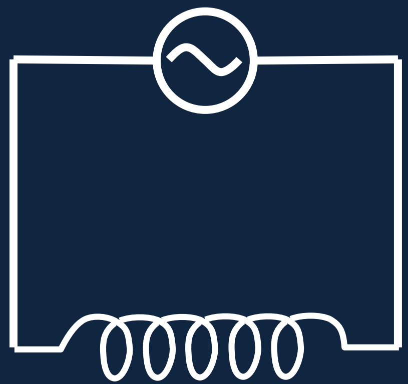
so $I = I_0 \cos \omega t$ where

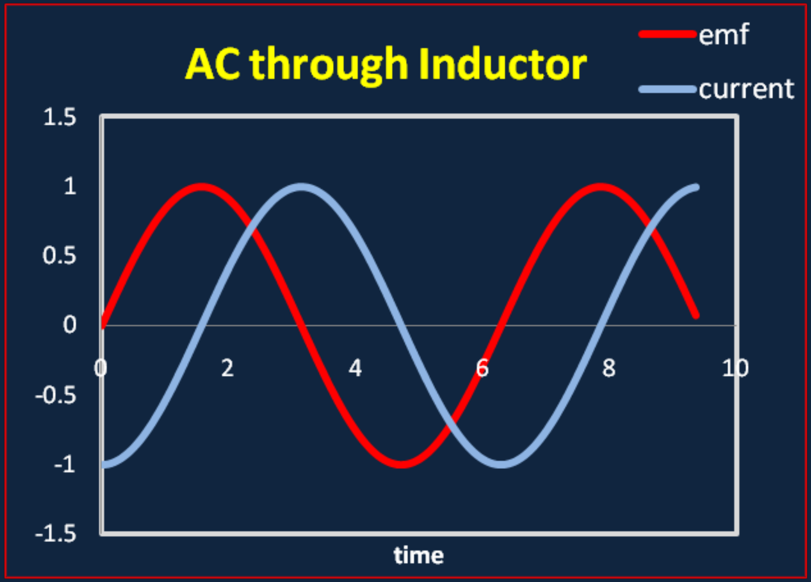
$$I_0 = V_0 / \omega L$$

ωL is the inductive reactance.

Power:

$$\overline{P} = \overline{IV} = I_0 V_0 \overline{\sin \omega t \cos \omega t} = 0$$



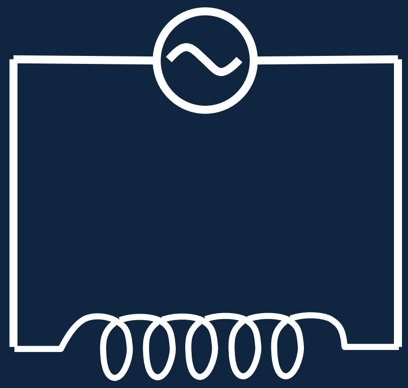


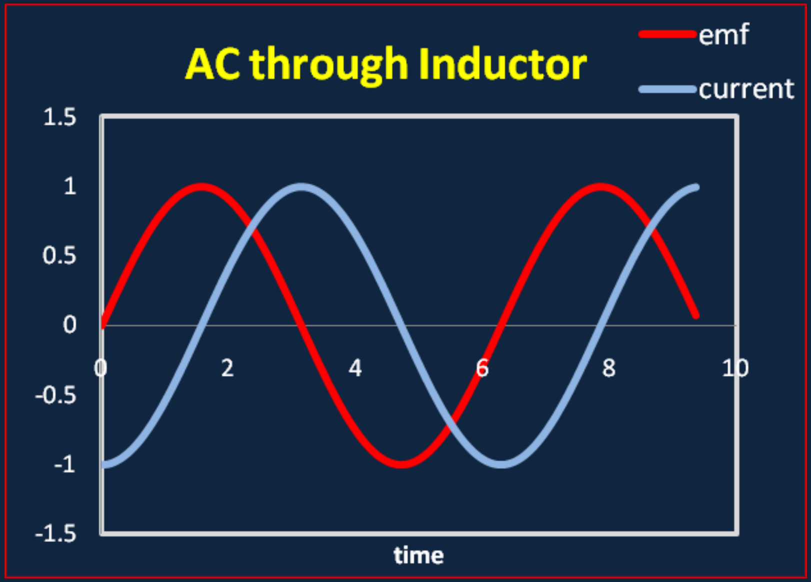
AC Source and Inductor…

$$I_0 = V_0 / \omega L$$

ωL is inductive reactance.

- Notice that this increases with frequency: faster oscillations mean more back emf.
- Note also that the peak in current occurs after the peak in voltage in the cycle.





AC Source and Capacitor

- For pure capacitance,

$$V_0 \sin \omega t = Q / C = \left(Q_0 \sin \omega t \right) / C$$

so

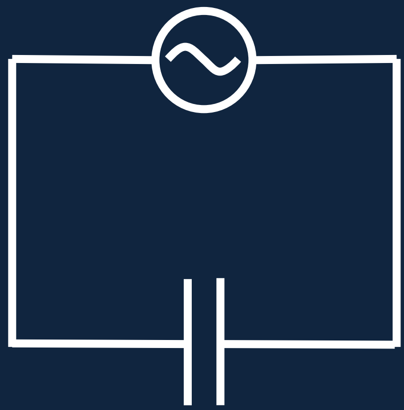
$$I = I_0 \cos \omega t = dQ / dt = Q_0 \omega \cos \omega t$$

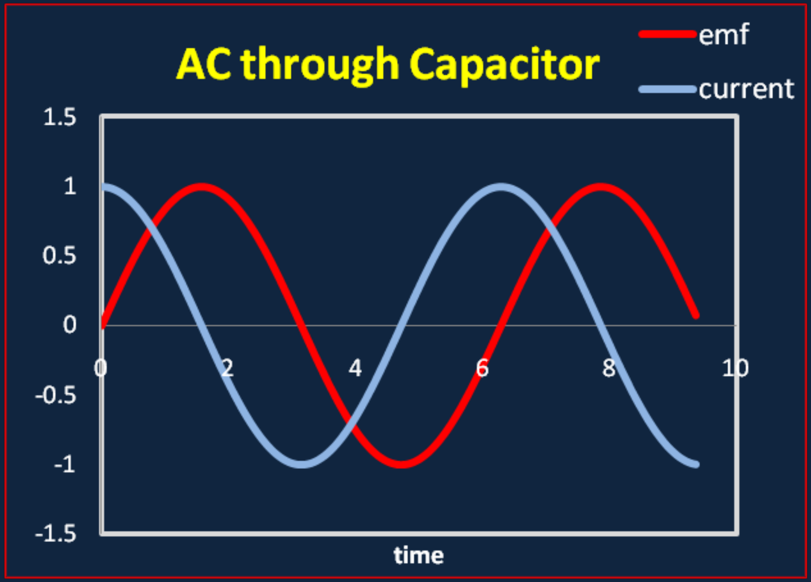
and from this we see that

$$I_0 = \omega C V_0$$

and the capacitive reactance is:

$$X_C = \frac{1}{\omega C}$$

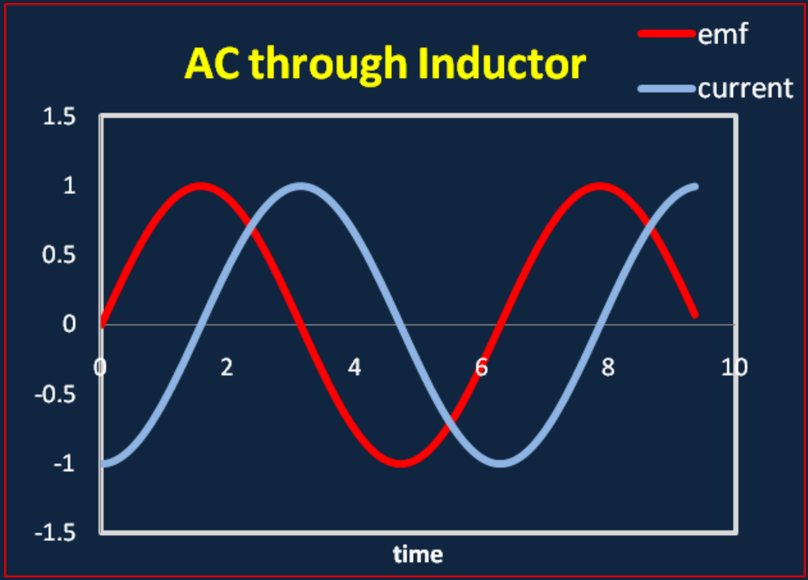


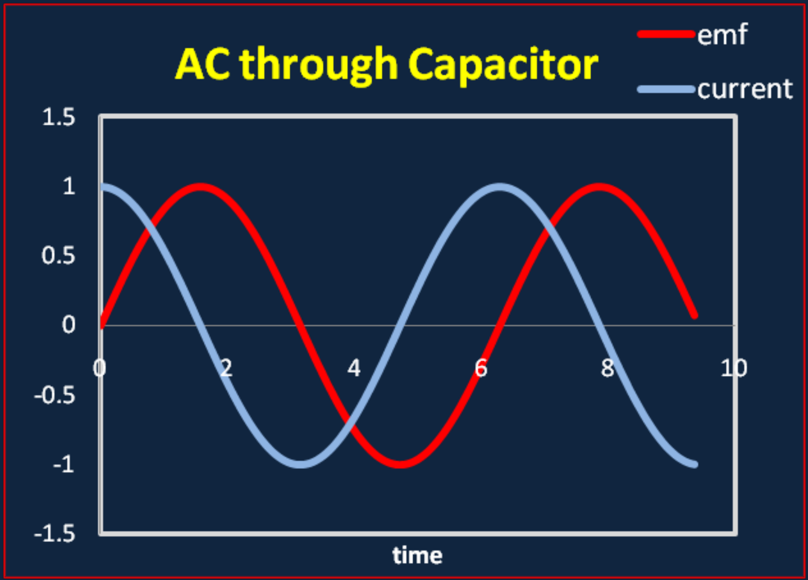


Comparing Pure L and Pure C

- For L, peak emf is before peak current, for C peak current is first.
- Mnemonic: ELI the ICE man.
- No power is dissipated in inductors nor in capacitors, since emf and current are 90° out of phase:

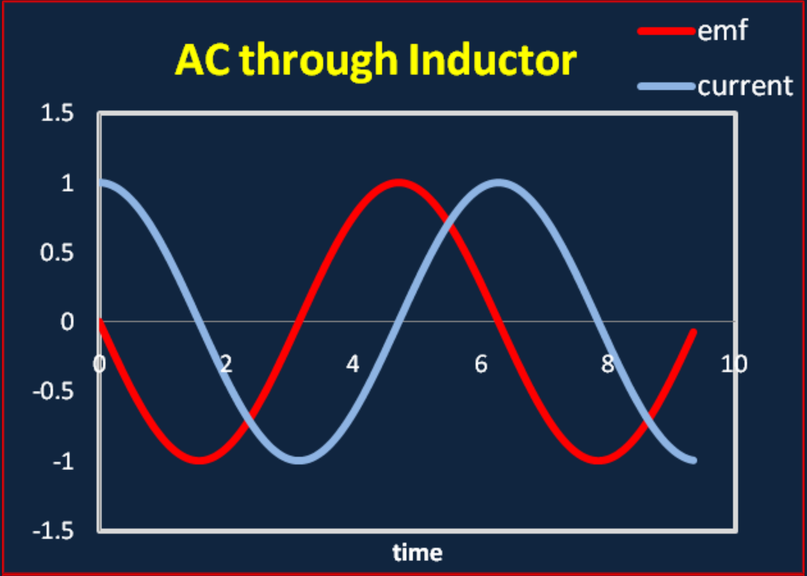
$$\overline{\sin\omega t\cos\omega t} = \tfrac{1}{2}\overline{\sin 2\omega t} = 0$$

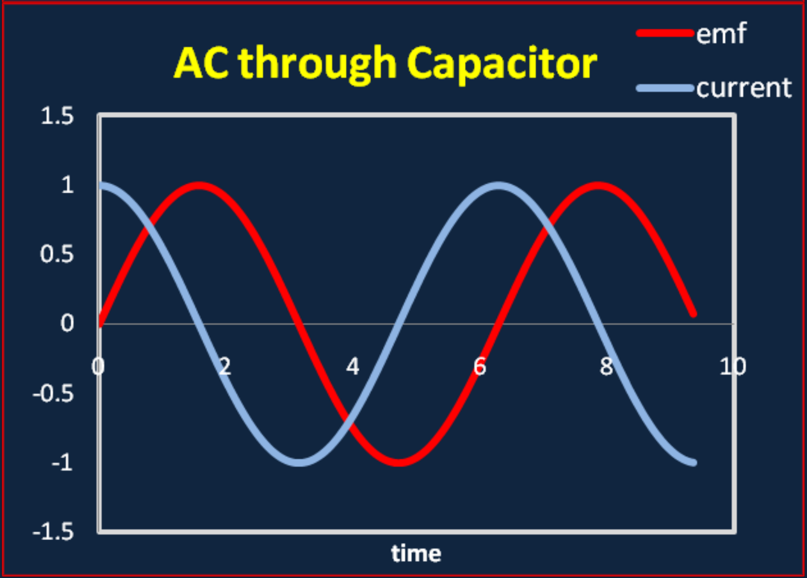




L and C in Series

- The same current is passing through both: the red curve is the emf drop over L and C respectively—notice they're in opposite directions!
- (We show here a special case $\omega = L = C = 1$ where no external emf is needed to keep current going—this is resonance.)





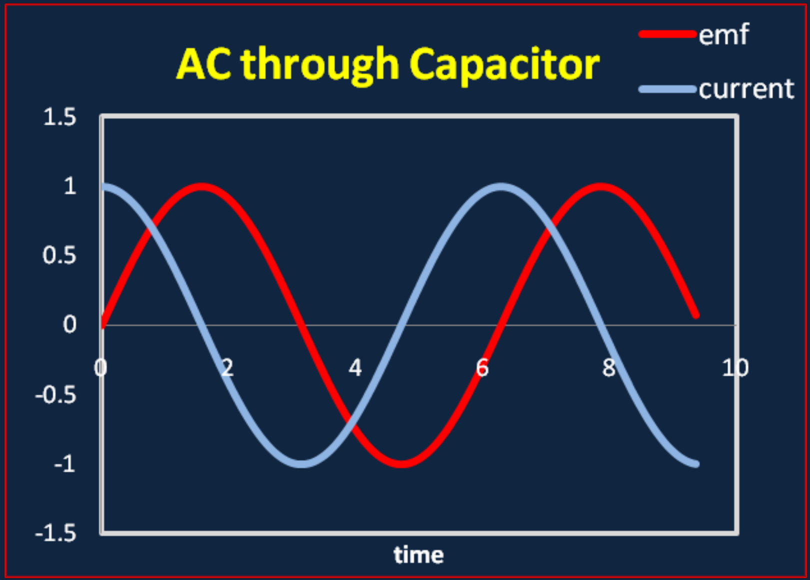
Clicker Question

- This shows ac emf and current for ω = C = 1.
- What happens to the current if ω is increased to 2, but emf kept constant?

A. Current doubles

B. Current halved

C. Current same maximum value, but phase changes.

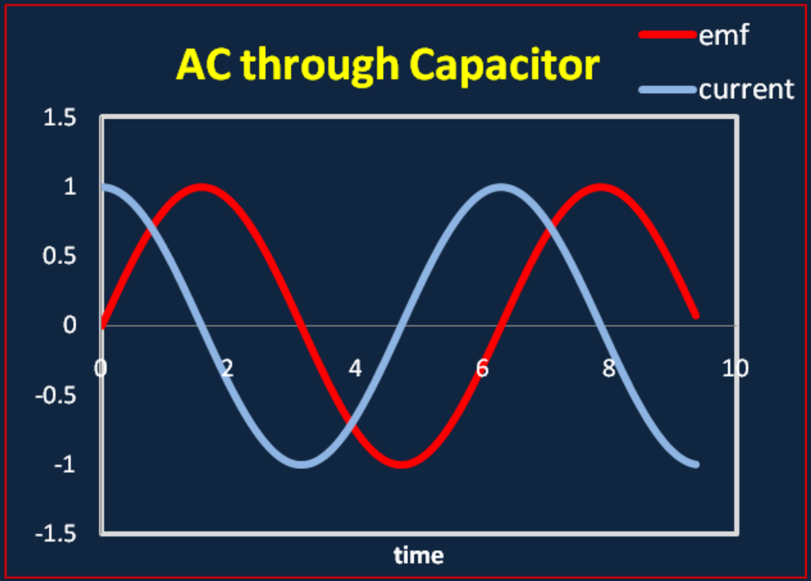


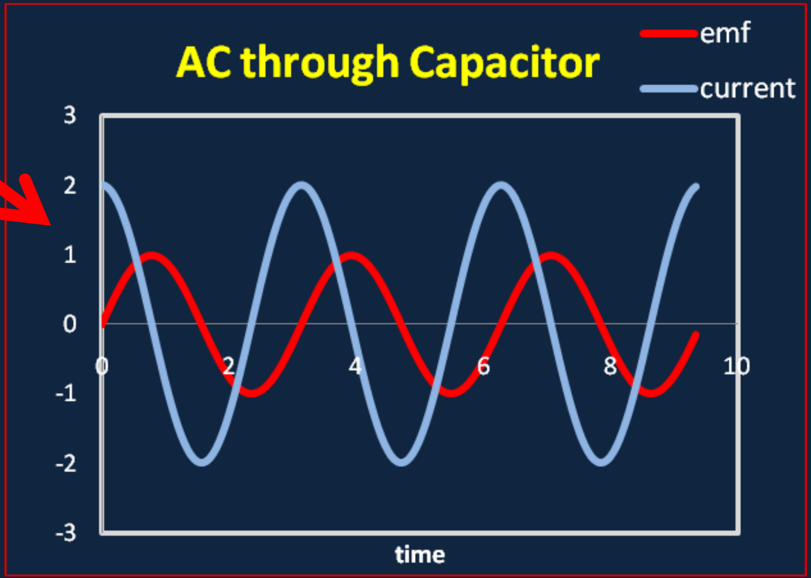
Clicker Answer

- This shows ac emf and current for ω = C = 1.
- What happens to the current if ω is increased to 2, but emf kept constant?

A. Current doubles

- Notice the axis is rescaled
- Capacitances pass higher frequency ac <u>more easily</u>—*opposite* to inductances!





Circuit with L, R, C in Series

- For a current of amplitude I_0 passing through all three elements, the emf drop across R is $I_0 R$, in phase with the current.
- Remember the emf drops across L, C have opposite sign—the total emf drop is $I_0(\omega L - 1/\omega C)$, but this emf is 90° out of phase.
- The current will therefore be ahead of the total emf by a phase angle ϕ given by:

$$\tan\phi = \frac{\omega L - 1/\omega C}{R}$$

Maximum emf and Total Impedance Z

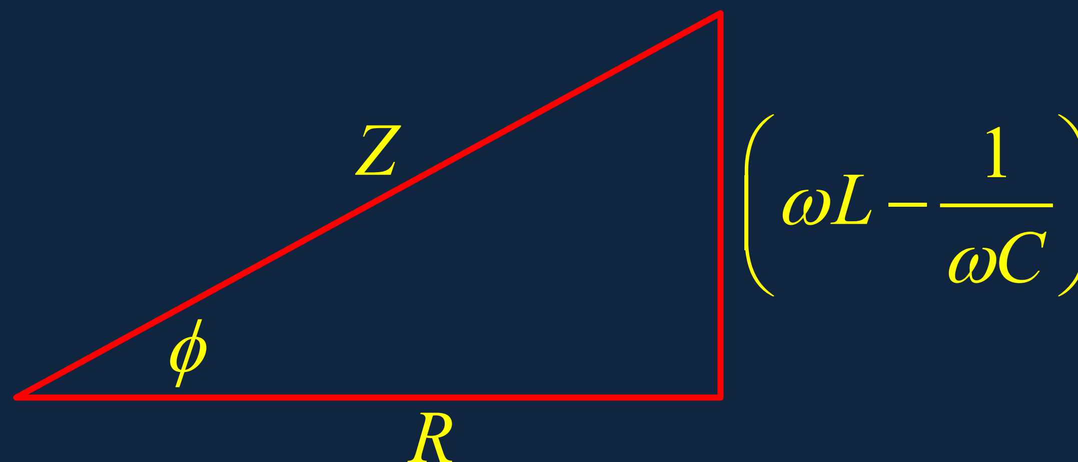
- For a given ac current, we find the emf driving it through an LCR circuit has two components which are 90° out of phase.
- To find the maximum total emf V_0, these two amplitudes must be added like vectors.
- The amplitudes are: I_0R, $I_0(\omega L - 1/\omega C)$.
- So

$$V_0 = I_0\sqrt{R^2 + \left(\omega L - \frac{1}{\omega C}\right)^2} = I_0 Z$$

Geometry of Z and ϕ

$$V_0 = I_0\sqrt{R^2 + \left(\omega L - \frac{1}{\omega C}\right)^2} = I_0 Z$$

The emf across the resistor is in phase with the current. The <u>total</u> emf is represented by Z, and if $\omega L > 1/\omega C$, the emf is ahead of the current by phase ϕ.



<u>Power dissipation</u> only in R: $\bar{P} = I_{\text{rms}}^2 R = I_{\text{rms}}^2 Z\cos\phi$

LCR Impedance Z as a Function of ω

$$V_0 = I_0\sqrt{R^2 + \left(\omega L - \frac{1}{\omega C}\right)^2} = I_0 Z$$

- Notice that if $\omega L = 1/\omega C$, $V_0 = I_0 R$, the minimum possible impedance. The capacitor and inductor generate emf's that exactly cancel. This is resonance.
- At very high frequencies, Z approaches ωL.
- At very low frequencies, Z approaches $1/\omega C$.

Spreadsheet link

Clicker Question

- Is it possible in principle to construct an LCR series circuit, with nonzero resistance, such that the current and applied ac voltage are exactly 90° out of phase?

A. Yes

B. No

Clicker Answer

- Is it possible in principle to construct an LCR series circuit, with nonzero resistance, such that the current and applied ac voltage are exactly 90° out of phase?

A. Yes

B. No ←

Because there is always energy dissipated, hence power used, in a resistor, and 90° out of phase means $\overline{P} = \overline{VI} = V_0 I_0 \overline{\sin \omega t \cos \omega t} = 0$.

Clicker Question

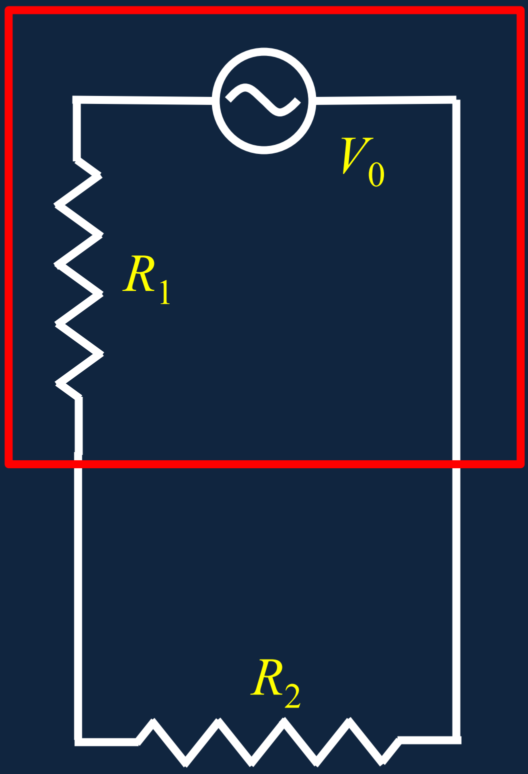
- *This is <u>for my information</u>: all answers will score 2.*

- Do you know the equation $e^{i\theta} = \cos\theta + i\sin\theta$?

A. Yes, I've covered it in a math (or other) course, and think I can probably work with it.
B. I've seen it before, but haven't really used it.
C. I have no idea what this equation is about.

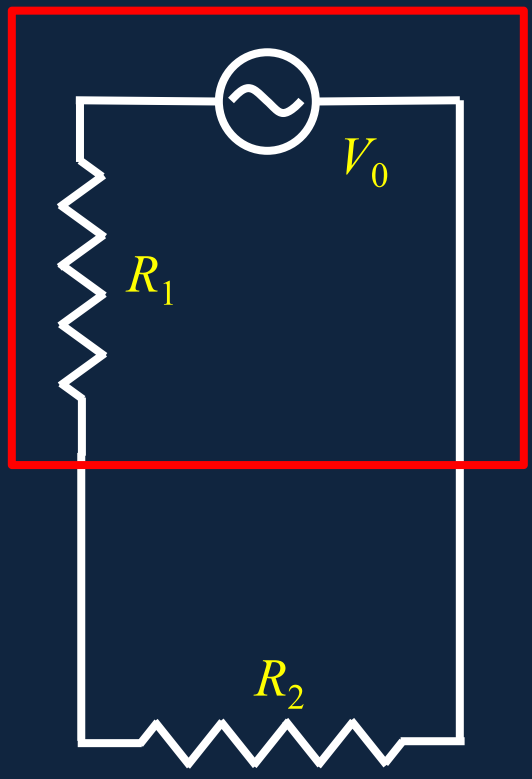
Matching Impedances

- A power supply (red box), say an amplifier, has internal resistance R_1, and neglibible inductance and capacitance. It generates an emf V_0.
- What speaker resistance R_2 takes maximum power from the amplifier?
- Power = $I^2 R_2, \quad I = \dfrac{V_0}{R_1 + R_2}.$



Matching Impedances

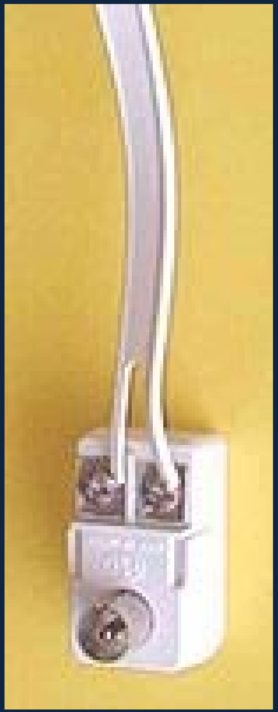
- Power $P = I^2 R_2, \quad I = \dfrac{V_0}{R_1 + R_2}.$
- So power $P = \left(\dfrac{V_0}{R_1 + R_2} \right)^2 R_2.$
- Notice this is small for R_2 small, and small for R_2 large.
- The maximum power is at $dP/dR_2 = 0.$
- You can check this is at $R_2 = R_1.$



Matching Impedances in Transmission

- Typical coax cable is labeled 75Ω, this means that the ratio $V_{\text{rms}}/I_{\text{rms}}$ for an ac signal, the impedance $Z = 75$.
- For the ribbon conductor shown, the corresponding impedance is 300Ω.
- Transmission from one to the other is done via a transformer such that the powers are matched $I_1^2 Z_1 = I_2^2 Z_2$.
- Therefore the ratio of the number of turns in the transformer coils is:

$$N_1 / N_2 = \sqrt{Z_1 / Z_2}.$$



Balun transformer