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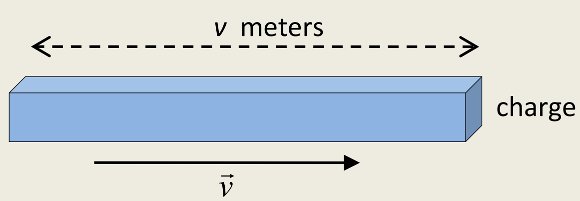
Physics 2415 Lecture 4: Gauss’ Law

Michael Fowler, UVa.

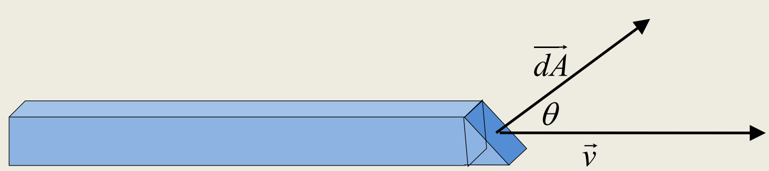
Electric Flux: a Watery Analogy

The main concept in Gauss’ Law is *electric flux*. What does this mean? The word *flux* just means *flow,* for example an influx of people into a room means they’re coming in. Before talking about electric flux, let’s look at something easier to visualize: flow of water.

We’ll begin by considering flow down a river. Suppose we stretch a net across the river, a fisherman’s net with thin strings and approximately square small holes, so that all the water flowing down the river goes through the net. For a steadily flowing river, the total flow through the net, in, say, cubic meters of water per second, doesn’t depend on whether the net is stretched flat across the river, or is curved by the current so that it bulges in a downstream direction—in either case, the total flow is all the water in the river. (I’m assuming here that the strings themselves are thin enough not to affect the flow measurably.)



If a fluid is flowing steadily at *v* meters/sec, the volume through an area *dA* perpendicular to the flow is *vdA* m^3/sec.



If the fluid is flowing through a small area *dA* having its normal at an angle θ to the direction of flow, the effective area seen by the flow is $dA\cos\theta$, the volume through the area is $vdA\cos\theta = \vec{v}\cdot\overrightarrow{dA}$ m^3/sec.

One way to find the total flow is to add the flows through all the little squares. We’ll assume the squares are small enough that the fluid velocity doesn’t vary significantly over one square: first, assume the little square is perpendicular to the direction of flow: if the square has area *dA* square meters (it’s small), and the fluid is flowing at speed *v* meters per second, then in one second a volume *vdA* cubic meters of fluid will flow through the square. But what if the square is *not* perpendicular to the flow? Then what counts is the *effective area* the flow sees—if the normal to the square is at an angle θ to the flow, this effective area is $dA\cos\theta$.

The standard notation is to represent the area by a vector $\overrightarrow{dA}$ of magnitude $dA,$ and direction perpendicular to the area, that is, along the normal. (The sign is of course ambiguous—we have to decide which way it's pointing on a case by case basis.)

Then the flow across the small area $\overrightarrow{dA}$ is $\vec{v}\cdot\overrightarrow{dA}$ (we have now chosen the vector $\overrightarrow{dA}$ to point downstream), and the total river flow F through all the holes in the net is

$$F=\int\vec{v}\cdot\overrightarrow{dA}.$$

It is important to realize that this total flow cannot depend on the detailed shape of the net: it must be the same for all nets that completely span the river, so that all the river water passes through the net.

Flow from a Point Source

To take a slightly different example, consider filling a large deep swimming pool using a hose, the end of the hose being deep in the water. We'll assume there are no currents present except the water flowing out of the end of the hose, and that this outflowing water goes out equally in all directions: this could be achieved, for example, by covering the end of the hose with a porous ball, so the water flows directly outwards from this ball (we'll ignore the obstruction presented by the incoming hose itself—suppose it's really thin).

Imagine now surrounding the source with a fishnet bag, a complete surface surrounding it, so all the water coming out the source goes through some hole in this fishnet. It's easy to see that if the hose is delivering *F* cubic meters per second, this will also be the total flow through the fishnet in a steady situation—water is not going to pile up inside the bag, it's incompressible for all practical purposes.

That is,

$$F=\int\vec{v}\cdot\overrightarrow{dA}$$

and this integral is the same for any closed surface surrounding the source.

At this point, we'll abandon the fishnet picture, and talk a little more abstractly about integrating over a surface surrounding the source, with the increment of area denoted by $\overrightarrow{dA}$ pointing outwards.

In particular, let's take a spherical surface of radius r surrounding the source. Since we've said the water is flowing out symmetrically in all directions, it will have the same speed $v(r)$ at all points on this centered spherical surface, and the flow vector will be parallel to the normal to the surface, so the total flow

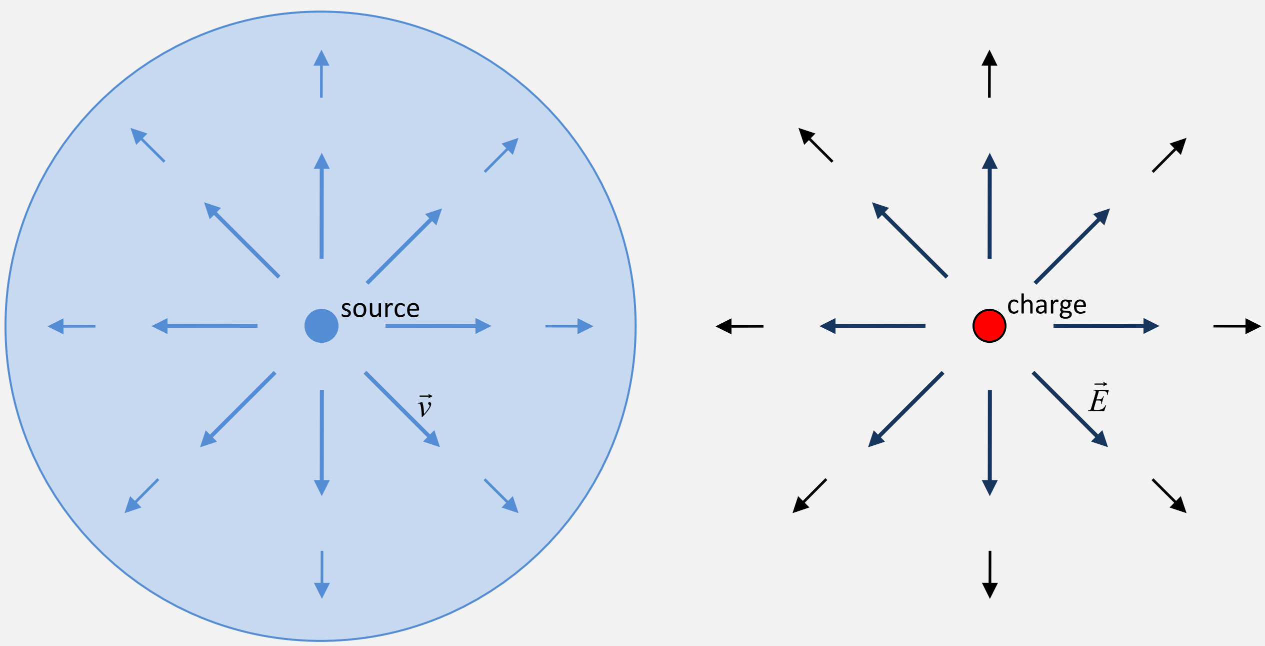
$$F=\int\vec{v}\cdot\overrightarrow{dA}=\int vdA=v\int dA=4\pi r^2v.$$

Therefore,

$$\vec{v}\left(\vec{r}\right)=\frac{F}{4\pi r^{2}}\hat{r}.$$

Notice this is formally identical to the electric field from a point source:

$$\vec{E}\left(\vec{r}\right)=\frac{Q}{4\pi\varepsilon_{0}r^{2}}\hat{r}.$$



For water flowing equally outwards in all directions from an underwater source, the vectors $\vec{v}$ denoting water velocity at each point in space form a pattern *identical* to that of the electric field vectors $\vec{E}$ from a point charge.

Note: Remember also this picture is a ***2D*** representation of a ***3D*** reality!

This is why historically people talked about "electric flux": the electric field vector from a point charge looks exactly like the fluid velocity vector for an incompressible fluid flowing symmetrically outwards from a small spherical source.

Specifically, the electric flux through a small area is defined in exact analogy with the flow of fluid through an area, it is just $\vec{E}\cdot\overrightarrow{dA}$, and the total electric flux through a closed surface with a single charge inside it is given by

$$\int \vec{E} \cdot \overrightarrow{dA} = Q / \varepsilon_0 .$$

We know the integral doesn't depend on which enclosing surface we choose, because the electric field vector is everywhere proportional to our water flow vector. We know the constant is Q / ε_0 because that's what we get if we take a spherical surface, with the charge at the center:

$$\int \vec{E} \cdot \overrightarrow{dA} = \int \frac{1}{4\pi\varepsilon_0} \frac{Q}{r^2} \hat{r} \cdot \overrightarrow{dA} = \frac{1}{4\pi\varepsilon_0} \frac{Q}{r^2} \int dA = \frac{Q}{\varepsilon_0}.$$

(The outward pointing unit vector $\hat{r}$ is parallel to the outward pointing little area vector $\overrightarrow{dA}$.)

We should mention that for a negative charge, the field lines of course point inwards: the fluid analogy is sucking the water out of the pool, a drain point.

What if we have a closed surface that doesn't include our point charge? What is $\int \vec{E} \cdot \overrightarrow{dA}$ in that case? The answer should be obvious from the flowing water analogy: if there is no source of water inside the surface, the water flowing in must balance the water flowing out in the steady state. That is to say, if there is no charge inside a surface in an electrostatic problem, then $\int \vec{E} \cdot \overrightarrow{dA} = 0$.

Gauss' Law for General Charge Distributions: Use Superposition!

We've given a detailed account of the value of $\int \vec{E} \cdot \overrightarrow{dA}$ over a closed surface for the field from a single charge, it's equal to Q / ε_0 if the charge is inside the surface, zero otherwise.

But it's easy to generalize, because any charge distribution can be represented as a (possibly large) number of point charges, and *the total electric field is the linear sum* of all the electric fields from these many point charges:

$$\vec{E}(\vec{r}) = \vec{E}_1(\vec{r}) + \vec{E}_2(\vec{r}) + \vec{E}_3(\vec{r}) + \vec{E}_4(\vec{r}) + \ldots$$

and therefore for a closed surface

$$\int \vec{E} \cdot \overrightarrow{dA} = \int \vec{E}_1 \cdot \overrightarrow{dA} + \int \vec{E}_2 \cdot \overrightarrow{dA} + \int \vec{E}_3 \cdot \overrightarrow{dA} + \int \vec{E}_4 \cdot \overrightarrow{dA} + \ldots$$

The first integral in the series will equal Q_1 / ε_0 if the charge Q_1 is inside the closed surface, zero otherwise. The same is true for all the terms in the series, so we conclude:

$$\int_{\text{closed surface}} \vec{E} \cdot \overrightarrow{dA} = (\text{total charge inside surface}) / \varepsilon_0$$

and this is **Gauss' Theorem**.