home

Physics 2415 Lecture 6: Gravitational and Electrostatic Potentials

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The Gravitational Analogy

As we've discussed, the gravitational field from a point mass and the electrostatic field from a point charge both go down with distance as $1/r^2$, and both fields satisfy the Superposition Principle. It might seem at first glance that electric fields are just going to follow the patterns set by gravitational fields—but of course, there's one huge difference! Electric charges can attract or repel, but there's no gravitational repulsion between masses.

You might think antimatter would repel matter, but experimentally it doesn't—all kinds of matter attract gravitationally. Then there's the so-called Dark Energy in the universe, which apparently causes everything to fly apart, *but* is only important on a cosmic scale. And, don't confuse *that* with Dark *Matter*, an as yet unidentified kind of matter we know must be there because its gravitational *attraction* is clear from the orbiting rate of stars in rotating galaxies, but it also has little effect on anything much smaller than a galaxy.

Near-Earth Gravitational Potential *mgh* and Its Electrical Equivalent

Let's begin by reviewing the Earth's gravitational field in this room. We can take it to be uniformly downward: a mass m will feel a downward force mg, and doubling the mass doubles the force. That is, the gravitational force on a mass m is $\vec{F}=m\vec{g}$ where $\vec{g}$ is a downward pointing vector of length g, the gravitational field strength.

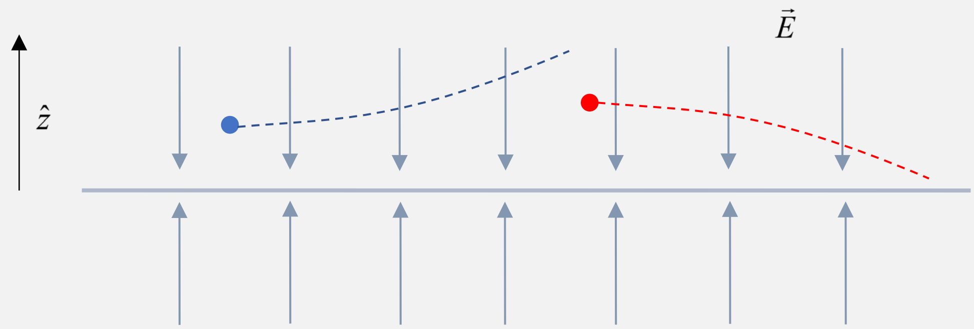
It takes work to lift a mass m from a point *A* to a higher point *B* against this gravitational pull: to be precise, as discussed earlier, it takes work $W=\int_A^B(-m\vec{g})\cdot\overrightarrow{ds}$, where $\overrightarrow{ds}$ is an incremental step on the path, and to move the mass at a steady rate we need to exactly counteract the gravitational force, that is we must exert a force $-m\vec{g}$, so the work done for the step $\overrightarrow{ds}=-m\vec{g}\cdot\overrightarrow{ds}$. Since this is a dot product, the only displacement that requires work is that in the upward direction, and it is easy to see that the total work done against the gravitational force on raising a mass m from *A* to *B* is $W=mg\left(h_B-h_A\right)$, where h_B is the height of point *B* above the ground. This work is stored by the system—it can be recovered simply by allowing the mass to fall back: it accelerates and gains kinetic energy equal to the work needed to raise it in the first place. That's why it's called "potential energy".

This leads naturally to the definition of a *gravitational potential*

$$U\left(h\right)=gh,$$

so $mU\left(\vec{r}\right)=mgh$ is a measure of the potential energy stored by a mass m as a function of position. Following normal usage, we denote height by h rather than z. There is of course the usual ambiguity concerning what "ground level" we take as $h=0$, but it is irrelevant in practice as we're always interested in potential energy *differences*.

The electrostatic analogy to gravity near the Earth's surface is the electric field in the region above an infinite, uniformly negatively charged insulating plane: *we covered this in lecture 3*.



Electric field from a uniform negative plane of charge, showing trajectories of charged particles.

The electric field has uniform strength and points towards the plane. The force on a charge *q* is $\vec{F} = q\vec{E}$.

Since there is no reason for this plane to be horizontal, we'll measure distance away from the plane as z, so $\hat{z}$ is a unit vector normal to the plane. By precise analogy with the gravitational discussion, the work needed to move a charge q along a path in this field is $W = \int_A^B \left(-q\vec{E}\right) \cdot \overrightarrow{ds}$, and, without further ado, we can define an electrostatic potential

$$V\left(z\right) = Ez = \left(\sigma / 2\varepsilon_0\right) z,$$

where σ is the charge density (this is for a charged plane, see diagram: remember that for a uniform layer of charge σ on the surface of a thick *conductor*, there will be *no factor two* in the denominator, because there is no field going into the conductor, all the field from the charge layer is on one side).

Now this was a negatively charged plane, so a positively charged particle projected upwards from this plane will follow a parabolic path and come back down, just as a mass will in the gravitational field in this room.

A *negatively* charged particle, on the other hand, will follow a parabolic path upwards! To see this, consider a particle projected parallel to the plane but some distance above it. Particles with the same mass but opposite charges will follow paths that are up-down mirror images of each other.

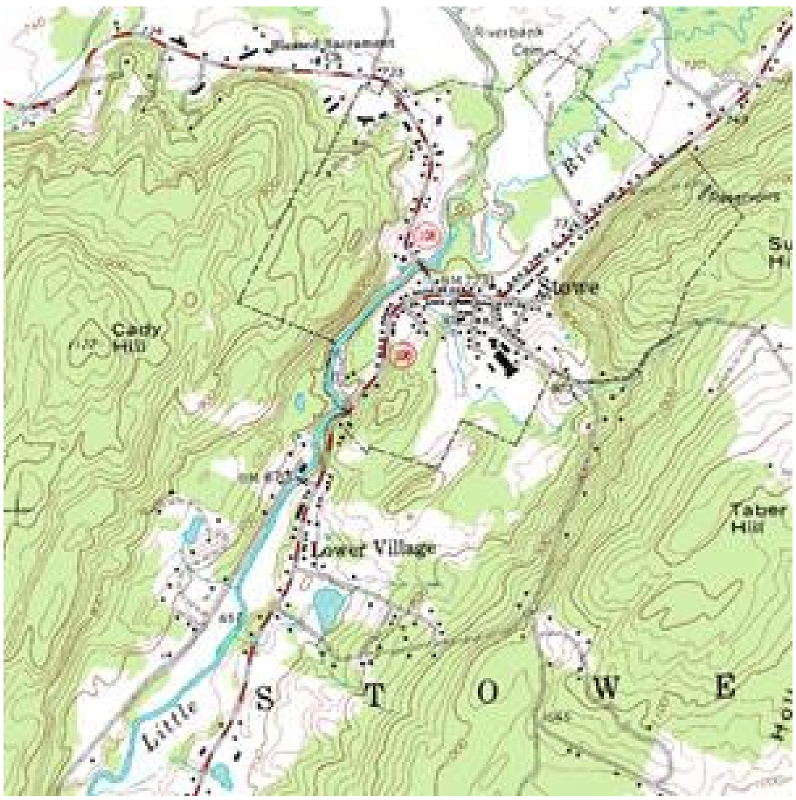
In practice, a uniform electric field as described above is well approximated in the space between two oppositely charged parallel planes. It is also a good approximation to the field near the surface of a charged conductor—near enough for the conductor to appear flat.

Gravitational Equipotentials are Contour Lines

Detailed maps of the countryside for hiking often include *contour lines* joining points at the same height h, in our language, points at the same gravitational potential. Walking along a contour line means you do no work against gravity. Of course, on level ground the force of gravity on you is balanced by the normal force from the ground, but if the ground is sloping and you walk uphill a distance $\Delta\vec{s}$ you do work against the component of gravity parallel to the ground, $-m\vec{g}\cdot\overrightarrow{\Delta s}=mg\Delta h.$ For a given speed, you work at the fastest rate (as of course you know!) by going straight uphill, meaning *perpendicular to the contour line*.

Notice from the map that the distance between contours is a measure of the steepness of the slope, an (inverse) measure of the strength of the gravitational field you are working against.

Simple example: for a map of a conical hill, the equipotentials would be concentric circles.



The Gravitational Analogy at Larger Distances

At distances comparable to the size of the Earth, the gravitational field has the familiar inverse square form $\vec{g}\left(\vec{r}\right)=-GM_E\hat{r}/r^2$.

The work done, and therefore the potential energy difference, on a path in this field is (as discussed above)

$$W=\int_A^B\left(-m\vec{g}\left(\vec{r}\right)\right)\cdot\overrightarrow{ds},$$

except that the gravitational field is no longer constant.

As before, the dot product denotes that *work is only done when there is displacement in the direction of the force*: here this means *displacement in the radial direction*, directly outwards. So, if *A* is at $\vec{r}_A$ from the center of the Earth, and *B* at $\vec{r}_B$, the gravitational potential energy difference for a mass $m=1$ is

$$U(\vec{r}_B)-U(\vec{r}_A)=GM_E\int_{\vec{r}_A}^{\vec{r}_B}\frac{\hat{r}\cdot\overrightarrow{ds}}{r^2}=GM_E\int_{r_A}^{r_B}\frac{dr}{r^2}=GM_E\left(\frac{1}{r_B}-\frac{1}{r_A}\right).$$

The very reasonable convention is to take the zero of gravitational potential energy to be at infinity, because in calculating total potential energies, we don't want to have to take account of stars in the next galaxy. This means that, outside the Earth's surface, the gravitational potential energy from the Earth's field is

$$U(\vec{r})=-\frac{GM_E}{r},\ \ r>r_E.$$

A mass resting at the Earth's surface has therefore a *negative* total energy (potential plus kinetic), a mass at rest far away has essentially zero total energy—so to get a mass away from the Earth it must be given a kinetic energy sufficient to get it up the potential hill: this corresponds to the escape velocity.

We are thinking in three dimensions and the equipotentials here are spherical *surfaces.*

Point Charges and Superposition

Switching now from gravity to the electrostatic analogy, the potential difference between two points in the field of a point charge Q (or outside a *spherically symmetric* charge distribution having total charge Q) is:

$$V_{\vec{r}_B}-V_{\vec{r}_A}=-\int_{\vec{r}_A}^{\vec{r}_B}\vec{E}\cdot\overrightarrow{ds}=-\frac{Q}{4\pi\varepsilon_0}\int_{\vec{r}_A}^{\vec{r}_B}\frac{\hat{r}\cdot\overrightarrow{ds}}{r^2}=-\frac{Q}{4\pi\varepsilon_0}\int_{r_A}^{r_B}\frac{dr}{r^2}=\frac{Q}{4\pi\varepsilon_0}\left(\frac{1}{r_B}-\frac{1}{r_A}\right).$$

In words, as with gravity, the potential difference is the work done against the field per unit charge/mass on moving from point *A* to point *B*.

For an actual *point* charge, assuming one could exist, it is clear that for small enough r the formula must break down (there cannot be infinite energies!) but even for electrons within atoms the formula is extremely accurate. (It does break down at electron scattering energies reached in particle accelerators: the field energy density becomes strong enough that from quantum mechanics virtual particle creation plays a significant role, this is termed *quantum electrodynamics*.)

Thus the potential in the electric field of a point charge is (taking it zero at infinity):

$$V(\vec{r})=\frac{Q}{4\pi\varepsilon_0 r}$$

Notice the sign!

If a positive charge is released in the field of a fixed positive charge, it will shoot away, and have kinetic energy far away. This is the *opposite* of the "escape velocity" scenario—that applies for a negative charge in the field of a fixed positive charge.

The *Principle of Superposition works for potential energies* just as it does for electric fields, since the potential energy difference is the sum of contributions from the different fields in the integral, so

$$V(\vec{r}) = \frac{1}{4\pi\varepsilon_0}\left(\frac{Q_1}{r_1} + \frac{Q_2}{r_2} + \frac{Q_3}{r_3} + \cdots\right)$$

and for continuous charge distributions, the sum becomes an integral.

An Atomic Energy Unit: the Electron Volt

For everyday life, the joule is a convenient unit of charge—one amp is a charge flow of one coulomb per second. Similarly, the volt, one joule per coulomb, is a convenient potential energy unit. But when we're analyzing energy transfer at the molecular level, the natural unit of charge is the electron charge (or minus it),