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# Physics 2415 Lecture 7: Field Lines, Equipotentials and the Dipole

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## Getting the Electric Field from the Potential

Important! It's usually easier to compute the potential than the electric field for a given charge distribution, since, for the field, one must sum over vectors. A very simple example is finding the potential on the axis of a uniform ring of charge—for a point on the axis, all the charge is at the same distance so no integral is necessary. You can go on (see next paragraph) to easily find the electric field component pointing along the axis (but the perpendicular field is tougher!)

So, suppose we have the potential as a function of position. How do we use it to get the electric field at a particular point $(x, y, z)$?

Write the formula for potential difference between two points separated by an infinitesimal distance $dx$:

$$V(x+dx, y, z) - V(x, y, z) = -\int_{(x,y,z)}^{(x+dx,y,z)} \vec{E} \cdot \overrightarrow{ds} = -E_x dx$$

from which

$$E_x = -\frac{\partial V(x, y, z)}{\partial x}$$

where the special derivative symbol means *partial* differentiation: $V$ is a function of three variables, but we're holding two of them constant—only allowing $x$ to vary.

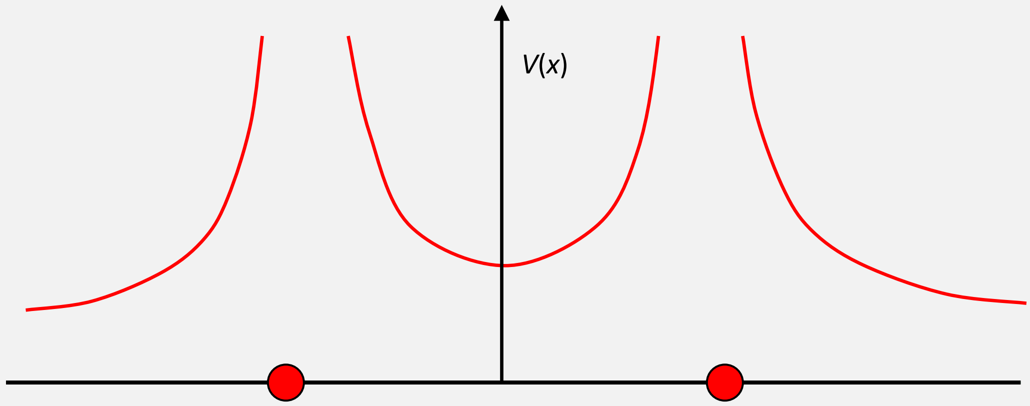
This formula, plus those in the other two directions, are often combined in a vector notation, written:

$$\vec{E} = -\vec{\nabla} V, \text{ or } \vec{E} = -\mathbf{grad}\, V.$$

In other words, the electric field in a particular direction is the negative of the slope of the potential in that direction: it's worth looking at a couple of examples to see this in action.

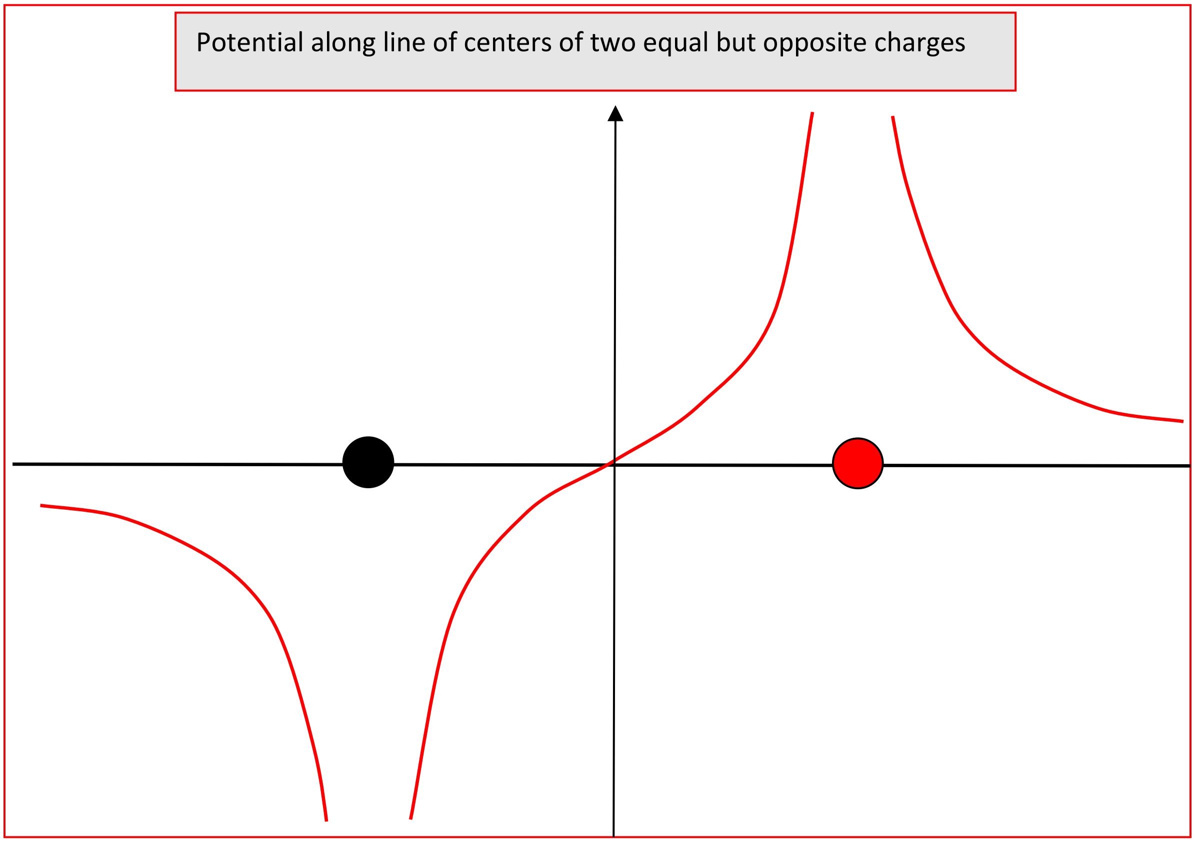
### Potential for Two Charges

First, consider two equal positive charges, let's say on the $x$-axis at $+a$ and $-a$, and think about the electric potential and electric field on the axis. This keeps it simple: the electric field points along the axis. The potential plotted along the *x*-axis looks like:




Potential from two equal positive charges plotted on line through center of charges

Over to the far right, the potential is sloping downwards, so the $\vec{E}$ field is pointing in the positive $x$-direction. Exactly half way between the charges, the potential bottoms out, that is, its slope is zero: so the electric field is zero at that point—not surprising, since a small positive charge there will be repelled equally by the two positive charges. In fact, the electric field changes sign (it's minus the potential slope) on going through that point. Note as well, though, that the value of the potential at that low point is *not zero*: if we moved away from that point in the $y$-direction, we'd be going downhill. (Check that by finding the electric field direction at a point on the $y$-axis.) The second example is a positive charge at $+a$, a negative charge at $-a$ on the $x$-axis. Now the potential along the axis looks like this:

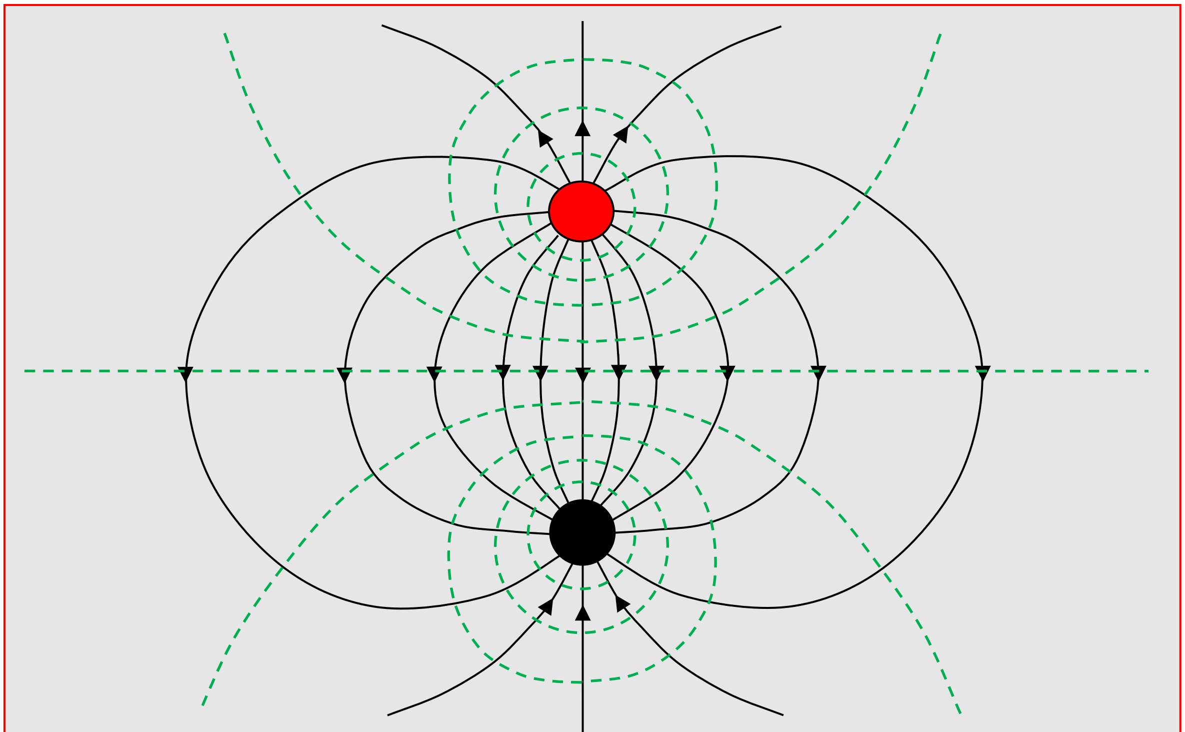



In this case, the electric field between the two charges is always strong and in the negative $x$-direction.

## Revisiting the Dipole: Field Lines and Equipotentials

If you sprinkle iron filings in the field from a magnet, they line up along the field direction, and you can draw what Faraday and Maxwell called "lines of force", parallel to the field at each point, to construct a picture of the field. We'll call them field lines (a more usual term).

We drew the field lines in lecture 3, now we'll add the equipotentials. Recall from the discussion above that they intersect at right angles, so, for example, for a pair of opposite charges we find:



Electric field lines of force and **equipotentials** for a **dipole**: charges *Q*, -*Q* a fixed distance $\ell$ apart

*Note*: Field strength and line spacing: this is a two-dimensional cut through a three-dimensional field. We can see where the field is strongest because the field lines are more concentrated. But this is tricky! We can draw field lines where we want. Furthermore, consider the field from a single small charged ball. If we draw the lines coming from the ball's surface equally spaced, then In this two-dimensional representation, the spacing between adjacent lines increases with distance proportional to $r$, suggesting the field strength goes down as $1/r$. But that isn't right, the lines are really coming out of the ball in three dimensions, and their density, the measure of field strength, actually goes down as $1/r^2$. So these two-dimensional representations of three-dimensional fields can be useful, but be careful—they're not quantitatively correct.

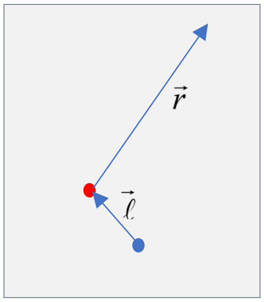
## Dipole Moments and Potential

Non-ionized molecules are overall electrostatically neutral (no net charge) but can have dipole moments, meaning the center of the positive charge is not the center of the negative charge. As in the diagram above (and previously mentioned in lecture 3, where we discussed a dipole in an external field) we can represent the dipole moment as equal but opposite charges $\pm Q$ separated by a vector distance $\vec{\ell}$.

The *dipole moment* is written

$$\vec{p} = Q\vec{\ell}.$$

The potential from the two charges at a point $\vec{r}$ measured from the positive charge as origin is




$$V(\vec{r}) = \frac{Q}{4\pi\varepsilon_0}\left(\frac{1}{|\vec{r}|} - \frac{1}{|\vec{r}+\vec{\ell}|}\right).$$

If we assume we are at a distance much greater than the size of the molecule $r \gg \ell$, we can approximate

$$|\vec{r}+\vec{\ell}| = \sqrt{r^2 + 2\vec{r}\cdot\vec{\ell} + \ell^2} \cong r\left(1 + \frac{\vec{r}\cdot\vec{\ell}}{r^2}\right),$$

and

$$\frac{1}{|\vec{r}+\vec{\ell}|} \cong \frac{1}{r}\left(1 - \frac{\vec{r}\cdot\vec{\ell}}{r^2}\right) = \frac{1}{r} - \frac{\vec{r}\cdot\vec{\ell}}{r^3}.$$

Putting these together, the potential at a point $\vec{r}$ from a dipole of strength $p = Q\vec{\ell}$ with $r \gg \ell$ is

$$V(\vec{r}) = \frac{Q}{4\pi\varepsilon_0}\left(\frac{1}{|\vec{r}|} - \frac{1}{|\vec{r}+\vec{\ell}|}\right) \cong \frac{Q}{4\pi\varepsilon_0}\frac{\vec{r}\cdot\vec{\ell}}{r^3} = \frac{1}{4\pi\varepsilon_0}\frac{p\cos\theta}{r^2}.$$

Notice the dipole electric *field* strength, $\vec{E} = -\vec{\nabla}V$, decreases for $r \gg \ell$ as $1/r^3$.

For large distances, if an *ionized* molecule has total charge $Q_{\text{tot}}$ the field strength very far away has dominant contribution $Q_{\text{tot}}/r^2$, this is sometimes referred to as the *monopole* field to distinguish it from the dipole field.

For general compact charge distributions, there are more terms, starting with the *quadrupole*, down by another factor of $1/r$ : think for example of charge $2Q$ at the origin and two charges $-Q$ displaced equally from the origin in opposite directions.