home

Physics 2415 Lecture 11: Microscopic Theory of Electric Current

Michael Fowler, UVa

Ohm's Law and Drude Theory

Ohm's Law, written down in 1825, relates the current I through a resistance R to the applied voltage V,

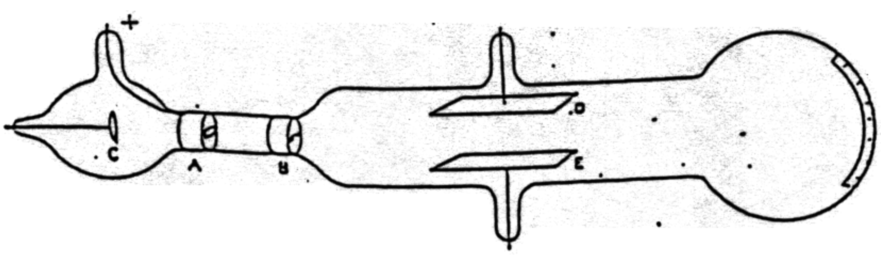
$$I = V / R.$$

He wrote this after extensive experimentation, finding it to be true for many metal resistances over wide temperature ranges (although R itself was usually temperature dependent).

Naturally, a simple law of such wide validity was a tempting target for theoretical speculation. Yet the first serious attempt to explain it with a model came seventy-five year later! Why did it take so long?

In fact, over this long period, no one had any idea what the "electric fluid" constituting the current looked like on a microscopic scale. In Ohm's Austria, for example, like other German speaking states, almost no one believed in the existence of atoms—many imagined solids to be continuous at all scales. One exception was Boltzmann (in the 1890's) who had read Maxwell's work on the kinetic theory of gases, and extended it, but had difficulty convincing his colleagues, and was told that on publishing he would not be allowed to mention "atoms". This negative reception probably contributed to his suicide.

The necessary conceptual breakthrough in understanding matter came seventy years later, in England, when J. J. Thomson (in 1897) discovered the *electron*. He was analyzing the rays in a cathode ray tube, here is his own diagram:



The "tube" is the overall glass enclosure, with a good vacuum inside. The cathode is marked C on the far left. In the experiment, the cathode is raised to a high negative voltage, the metal rings A and B (see figure) are positive. At high enough voltage difference, rays were seen to emanate from C, some passed through slits in A and B, and struck the right-hand end of the apparatus, where the glass glowed. The effect is much enhanced by coating the glass at that end with a phosphor which shines brightly when struck by the rays.

Next, (see figure), Thomson charged plates B and C oppositely so the rays passed through a sideways electric field. The rays were deflected! *They must be charged particles*. As we'll soon discuss, measuring the deflection for given electric field strength, and also measuring deflection by a magnetic field, a simple calculation yielded the ratio charge/mass. Assuming the charge equaled that of the hydrogen ion in magnitude (as was soon established), their mass was 2,000 times smaller than the lightest atom.

These must be the electric fluid! And, they must have been inside atoms in the cathode. Thomson repeated the experiment with different materials making up the cathode. He always found the same particles emitted. Evidently these "electrons" were present in all atoms.

Thomson then suggested the *plum pudding* model of the atom: a sphere of positive charge with electrons embedded like raisins in a pudding. Electrons could be knocked out, leaving a positively charged ion—the electron had the right charge value for this to work.

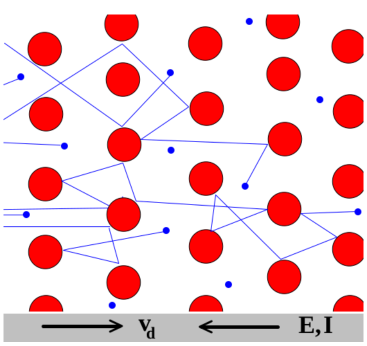
A crystalline material like copper was known to have a regular array of atoms, and it was known that some solids, like salt, NaCl, form by the atoms becoming ionized and electrostatic attraction binds them into a cubic grid, where the nearest neighbors of an ion all have the opposite sign charge. In the pudding picture, this could be understood by saying an electron leaves each Na and becomes part of a neighboring Cl.

But Cu must be a little different from NaCl. For one thing, it conducts electricity. This suggests that as the atoms bind to form the solid, some fraction of the electrons remain unattached, free to move through the crystal—and these become a current when an electric field is applied.

In 1894, Paul Drude became a top professor at the University of Leipzig, having gained a Ph. D. studying the diffraction of light by crystals, most naturally understood in terms of scattering from a regular array of atoms or ions. This was also the time when Maxwell's theories were becoming more widely accepted in Germany, both his electromagnetic equations and his kinetic theory of gases.

So by the late 1890's *for the first time* concepts were available to formulate a semi-plausible theory of electrical conduction: Maxwell's theory of gases (extended by Boltzmann) could be applied to the "gas" of electrons, including Maxwell's predicted temperature-dependent velocity distribution. Of course, Maxwell's theory was for gas molecules in an otherwise empty box. Hopefully the velocity distribution would still more or less work for the electrons in a "box" already containing the rows of ions, even though the electrons must keep colliding with the ions? Also, the electrons repelled each other, but, at least on average, this was compensated by the background attraction from the ions.

Anyway, despite these obvious objections, Drude hypothesized that, statistically, the electron gas had the symmetric Maxwell velocity distribution when in zero external electric field, although each individual electron had a probability dt/τ of scattering in time dt, and would scatter to some other random velocity in the Maxwell distribution, with no memory of its velocity just before scattering.



Now switch on an electric field $\vec{E}$ at $t=0$. Each electron will accelerate to gain velocity $\vec{v}=e\vec{E}t/m$ (added to its original Maxwell distribution velocity) until it collides with an ion, at which point the process will repeat.

Suppose at some later time t we take a freeze frame shot of the electrons: with the given scattering rate, the average time interval since the last scattering is τ, so the *average* electron velocity (called the *drift velocity*) is

$$\vec{v}_{\mathrm{d}}=e\vec{E}\tau/m.$$

(Remember the pre-collision velocities average to zero.)

Taking the density of electrons to be n, the electric current density

$$\vec{j} = ne\vec{v}_{\mathrm{d}} = \frac{ne^2\tau}{m}\vec{E}.$$

Defining the conductivity

$$\sigma = \frac{ne^2\tau}{m},$$

we have Ohm's law in the form

$$\vec{j} = \sigma\vec{E},$$

relating the current density to the electric field strength.

As we discussed in the last lecture, to recover the traditional form $I = V / R$ for current down a wire, take the wire to have cross-section area A and length ℓ, so the total current $I = jA$, assuming it's uniform across the area, which it is, and the voltage drop down the piece of wire $V = E\ell$ so

$$j = I / A = \sigma V / \ell,$$

and

$$R = V / I = \ell / A\sigma.$$

Standard notation is to define the *resistivity*

$$\rho = \frac{1}{\sigma},$$

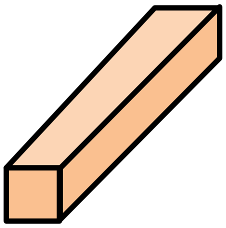
so the resistance R of a length ℓ of wire having cross-section A is

$$R = \frac{\rho\ell}{A}.$$

Checking Drude's Model Against Experiment

First, it does predict Ohm's law. But to get some picture of how it relates to reality, it would be useful to find, for example, the scattering time τ.

By 1900, the charge e and mass m of the electron were known, as was the electron density n, at least approximately, so the drift velocity could be measured as follows:



Take a piece of copper wire, say 1mmx1mm cross section, 1m long carrying 5 amps.

This is 1cc of Cu, about 10 gms, about 10^{23} conduction electrons (assuming one per atom), about 15,000 Coulombs of electron charge.

Therefore, at 5 amps (C/sec) it takes 3000 secs for an electron to drift 1m.

Bottom line: the drift velocity is of order 0.0003 m/sec.

This wire has resistance $R = \rho\ell / A \approx 0.02\Omega$ so from Ohm's law $E \approx 0.1\,\text{V/m}$.
This field will accelerate the electrons, $ma = eE$, approximate acceleration = 2×10^{10} m/s ^{2} This reaches the drift velocity in about 0.5×10^{-14} seconds, that must be the time τ.

So how far does the electron move on average between collisions? (This is the *mean free path*, often labeled ℓ.)

Drude assumed the electrons had a Maxwell velocity distribution, which would give an average velocity $\bar{v}$ at temperature T:

$$\tfrac{1}{2}m\bar{v}^2 = \tfrac{3}{2}k_B T,$$

with k_B Boltzmann's constant. This gives an average electron speed of order 10^5 m/sec, so the distance between collisions, the mean free path, is of order 0.5×10^{-9} m, close to the interionic distance, a reasonable sounding result.

Other insights from Drude's theory included the behavior of a current when a magnetic field was added (the Hall effect), and the previously mysterious relationship between electrical conductivity and heat conductivity (in metals heat is mainly conducted by electrons).

What About Quantum Mechanics?

However, it turned out that although the picture of an electron gas with random scatterers was essentially correct, the advent of quantum mechanics changed everything. An electron in a regular crystal is wavelike, and passes through a perfect crystal at zero temperature without scattering, like light through glass. It *is* scattered by impurities, and by thermal vibrations. This explains why resistivity of metals increases approximately linearly with temperature over a wide range. The formula is:

$$\rho_T = \rho_0\left[1+\alpha\left(T-T_0\right)\right].$$

An old incandescent (not LED) bulb has a tungsten wire at about 3300K, and α = 0.0045, from which the resistivity is not far off being proportional to absolute temperature.

Experimentally (and theoretically) the electron mean free path is at least an order of magnitude more than Drude's model suggests, yet the mean free *time*, which we found from the drift velocity, must still be the same. Going back to a particle viewpoint, this means the electrons are going at least an order of magnitude faster than Maxwell's velocity distribution predicts. Turns out they are not at all like a classical gas. First, the wavelike nature means that, as in the Bohr atom, only certain wavelength are allowed (so a whole number of wave oscillations fit in the box), and, second, there's the exclusion principle: no two electrons can be in the same quantum state. This means the free electrons are forced up into high velocity states—this is the essential point. Unfortunately, a fuller explanation would take several lectures— meaning a proper introduction to quantum mechanics.

AC and DC

We'll discuss AC in much more detail later.

Batteries provide direct current, DC: it always flows in the same direction.

Almost all electric generators produce a voltage of sine wave form:

$$V = V_0 \sin 2\pi ft = V_0 \sin \omega t.$$

In a resistance R this drives an alternating current, AC,

$$I = \frac{V_0 \sin \omega t}{R} = I_0 \sin \omega t$$

and power

$$P = VI = I^2 R = I_0^2 R \sin^2 \omega t = \left(V_0^2 / R \right) \sin^2 \omega t.$$

So the power is rapidly oscillating, what matters in practice is almost always the average power.

The average value $\overline{\sin^2 \omega t} = \frac{1}{2}$ (from $\overline{\sin^2 \omega t} + \overline{\cos^2 \omega t} = 1$.)

We define the *root mean square* voltage V_{rms} by

$$V_{\text{rms}} = \sqrt{\overline{V^2}} = V_0 / \sqrt{2},$$

so the average power

$$\overline{P} = V_{\text{rms}}^2 / R.$$

The standard 120 V AC power is $V_{\text{rms}} = 120V$, so the maximum voltage $V_0 = 120\sqrt{2} \approx 170$ V.

Semiconductors

In the Bohr model of the hydrogen atom, an electron circles around a proton.

An n-type semiconductor is a dielectric insulator which has been doped—atoms having one more electron than the insulator atoms are scattered into it.

The extra electron circles the dopant atom, but is loosely bound because the dielectric shields the electric field—it looks like a big Bohr atom. As the temperature is raised, these electrons break away from their atoms, and become available to conduct electricity.

Bottom Line: Conductivity *increases* with temperature.

Superconductors

A superconductor has exactly zero resistivity.

In 1911, mercury was discovered to superconduct ($R = 0$) when cooled below 4K.

Superconducting magnets are widely used, in MRI machines, etc.

There are now materials superconducting above the boiling point of liquid nitrogen, making long distance transmission lines feasible.

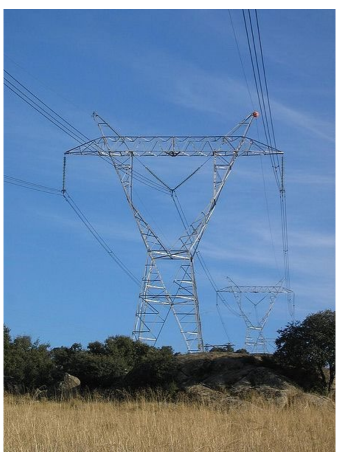
Superconductivity is a quantum phenomenon.

Why Bother with AC? (*we'll discuss this more later*)

Because, as we'll see, it's very easy to transform from high voltage to low voltage using transformers.

This means for long distance transmission we can use very high voltage, hence small currents and thinner wires, but transform to less dangerous low voltages for local use.

Long distance lines use aluminum wires. Copper is a better conductor, but is much heavier and more expensive. Steel is sometimes added for strength.



Sometimes DC is used for a single long line.

- This 3 gigawatt DC line (enough for 2 to 3 million households) transmits hydropower from the Columbia river to Los Angeles.
- At these distances, it gets tricky synchronizing the phase of AC power.


Washington
Celilo
Oregon
Pacific
Alternating
Current
Intertie
Pacific
Direct
Current
Intertie
Nevada
California
Sylmar