Physics 2415 Lecture 12: DC Circuits I

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Introduction: Electromotive Force and Terminal Voltage

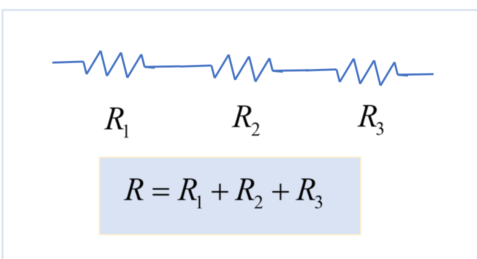
In this lecture, we'll analyze current flow in a network of resistances and include the possibility of batteries in some branches. We only address steady current flow, so do not include capacitances or inductances—these will be dealt with a little later. Beginning with the simplest case of a single battery, the potential difference that drives current originates in the chemical reactions inside the battery, at the surface of contact of the electrolyte and the terminals, called the anode and cathode. The two chemical reactions (releasing an electron at the anode to go around the circuit to combine chemically at the cathode) add to give a driving potential called the electromotive force, denoted by $\mathcal{E}$. This drives the current around the circuit but also through the battery itself, which has its own resistance, usually denoted by r. Thus the potential delivered outside, called the terminal voltage and denoted by V, is given by

$$V = \mathcal{E} - Ir.$$

Often r is small enough for this correction to be ignored.

Remark: don't worry too much about the names anode and cathode. Check the Wikipedia article. For one thing, the names are switched on recharging. Also, in vacuum tubes the heated element is always called the cathode. Just concentrate on how the electrons/ions are moving.

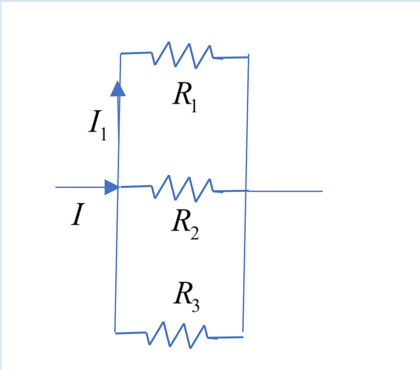
Resistances in Series and Parallel



Applying Ohm's Law $V = IR$, the same current passes through all three resistances, so there are successive voltage drops IR_1, IR_2, IR_3 for a total voltage drop

$$V = IR_1 + IR_2 + IR_3 = IR$$

where $R = R_1 + R_2 + R_3$. Resistances in series just add.



$$I = I_1 + I_2 + I_3 = \frac{V}{R_1} + \frac{V}{R_2} + \frac{V}{R_3}$$

$$I = \frac{V}{R}$$

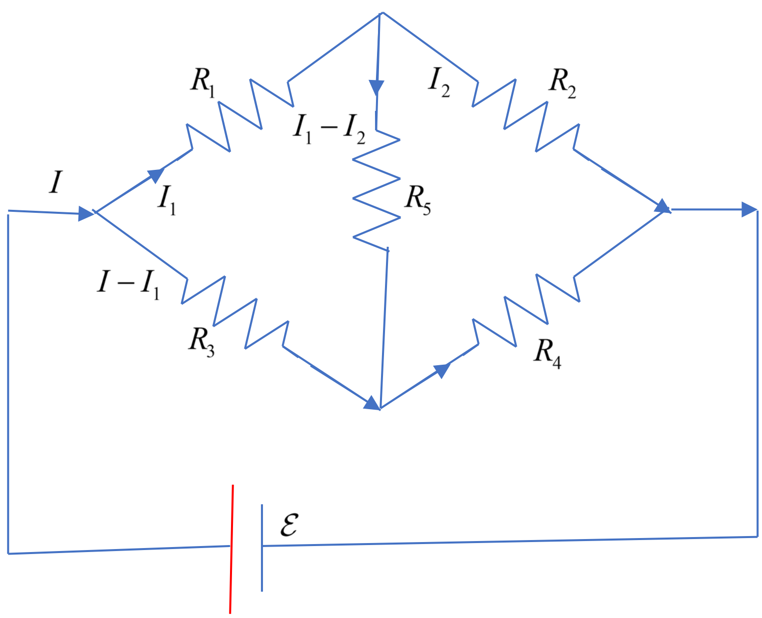
Parallel resistances all have the same voltage drop, the total resistance (see figure) is given by

$$\frac{1}{R} = \frac{1}{R_1} + \frac{1}{R_2} + \frac{1}{R_3}.$$

This is more obvious thinking in terms of the conductance (the inverse of the resistance): conductances just add, like parallel pipes conveying water.

More General Networks: Kirchhoff's Laws

We first consider a network of connected elements, as in this diagram, the individual elements can be resistances or batteries. (We'll add capacitances and



inductances later.)

To analyze such a network, we label each element with its resistance R_i, the current I_i and the emf of any battery in that element $\mathcal{E}_i$. Then we use Kirchhoff's laws.

Kirchhoff Law #1: Junction Rule. At any connection point between elements, the total ingoing current must be zero.

In other words, charge cannot be piling up—the junction has no capacitance.

We have already applied this rule in the above diagram to reduce the number of unknown currents from five to two. The currents must be labeled with a value I_j and an arrow indicating direction.

Kirchhoff Law #2: Loop Rule. The potential drop across an element is $I_j R_j$. (plus possible battery term).

The total potential change on going round a closed loop back to the same point must be zero. The electric field is conservative, so

$$\sum_{\text{loop}} I_j R_j = 0.$$

If you take a walk on a hillside and finish at the same spot you began from, your total change in gravitational potential is zero. This is the same thing.

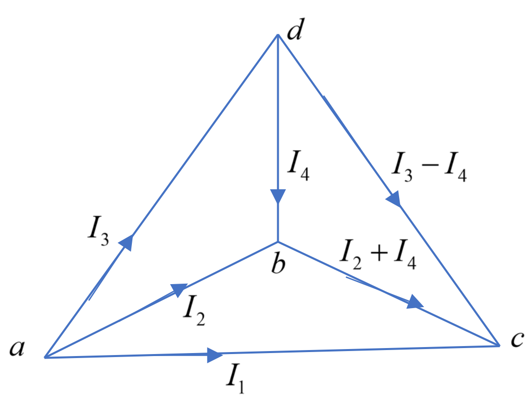
General Strategy for Solving Resistance Networks:

First, notice if there are resistances in series that can just be added, or in parallel that can be combined. (There may not be any.)

Second, label the current through each resistance, taking full advantage of the junction rule to minimize the number of unknowns.

Third, apply the loop rule to generate a number of equations equal to the number of unknown currents.

Solve these simultaneous linear equations to find the currents. You can then use Ohm's law to find the voltage drop for any resistance.



An Example: Compute the resistance R_{ac} of this network from a to c given that all lines are one ohm resistors except dc, which has resistance r.

In drawing the diagram, we've already applied the Junction Law at b, d to avoid introducing yet more unknown currents. This should always be done.

The total current flowing from a to c is

$I = I_1 + I_2 + I_3$.

Call the resistance of the network from a to c R_{ac}, then $V_{ac} = IR_{ac} = I_1$, the last being the voltage drop across the one ohm resistor ac.

So $R_{ac} = I_1 / I$.

Now we add to zero the voltage changes on going around loops, using $V = IR$ for each element, with $R = 1$ except for dc where $R = r$.

We have four unknown currents, but already have one equation above, given the external current I, so we need three loop equations. Here they are:

Loop abd: $I_2 = I_3 + I_4$. Loop abc: $I_1 = 2I_2 + I_4$. Loop acd: $I_1 = (1 + r) I_3 - rI_4$.

From the first two, $I_1 = 2I_3 + 3I_4 = (1+r)I_3 - rI_4$, so $(r-1)I_3 = (3+r)I_4$.

It is now straightforward to express all currents as multiples of I_4 (do it!) to find

$$R_{ac} = \frac{I_1}{I_1 + I_2 + I_3} = \frac{3+5r}{8(1+r)}.$$

Exercise: Consider the three special cases $r = 0, 1, \infty$. See if you can find an easy way to find R_{ac} for each of these three cases, without going through all the work above.