Physics 2415 Lecture 22: Mutual Inductance

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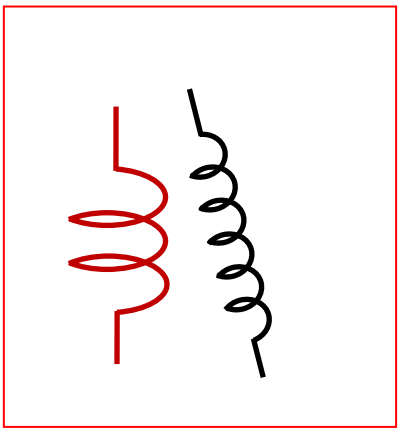
Definition, and an Important Symmetry

We've already discussed how an ac transformer operates: the alternating current in the first coil generates an oscillating magnetic field, the soft iron guides all the magnetic flux through a secondary coil, where the changing flux generates an ac emf, which can be tapped off as a power source. In operation, the magnetic flux from both coils threads both coils, and a complete analysis requires solving coupled equations. But since both coils experience the same changing flux, it follows immediately that the emfs thereby generated are in the ratios of the number of turns, and since the emf in the primary is just balancing the supplied external emf, voltage in secondary/voltage across primary = N_2/N_1. Neglecting the tiny losses from Joule heating (including eddy currents), the power absorbed in the primary must be the power supplied by the secondary, so the ratios of the currents will be the *inverse* of the voltage ratios.



More generally, if two coils 1, 2 are in proximity, a current through one will cause some magnetic flux to thread through the other, so a changing current in one will induce an emf in the other. From the Biot-Savart Law, the magnetic field from a current is linear in the current (directly proportional to it) so from Faraday's law, the induced emf will be linear in the rate of change of the current I_1 in coil 1.

The coefficient of proportionality is called the **mutual inductance**, and is denoted by *M*:

$$\mathcal{E}_2 = -M_{21}\frac{dI_1}{dt}.$$

Putting in the minus sign is standard practice—as usual, the direction of the emf and consequent current should be found using Lenz' law.

The mutual inductance can also be expressed purely in terms of the magnetic flux linkage:

it's just the total magnetic flux through coil 2 when there is unit current in coil 1.

Writing this total flux as $\Phi_{I_1=1}$, for current I_1 in coil 1 the total flux through coil 2 is $\Phi_{I_1=1}I_1$, and for a changing current in coil 1 the induced emf in coil 2 is given by

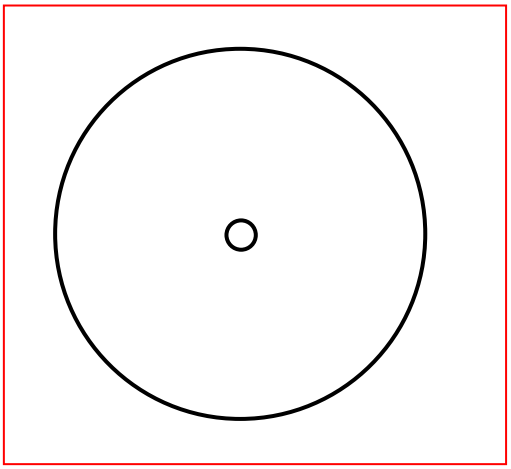
$$\mathcal{E}_2 = -\frac{d\Phi}{dt} = -\Phi_{I=1}\frac{dI_1}{dt}$$

It turns out that for two coils, or indeed for any two current-carrying conductors, the **mutual inductance is symmetric**:

$$M_{12} = M_{21}.$$

This is by no means obvious! It cannot be proved using simple arguments—it is necessary to use vector calculus. Recall that the electrostatic potential could be written in terms of a potential. It turns out that the magnetic field can be written as the curl of a vector potential, and this formulation plays an essential role in the proof of symmetry of mutual inductances.

This symmetry can be handy! Quite often, it's easy to evaluate the inductance one way, but not the other. For example, consider trying both approaches when coil 1 is a long solenoid going through a single loop, this loop being coil 2.



Another example: consider a small circular loop at the center of a large circular loop, both in the same plane. Putting a current I in the big loop gives a field on the axis at the center of $\mu_0 I / 2R$, so if the small loop has *r* << *R*, we have

$$M_{12} = \mu_0 \pi r^2 / 2R.$$

Notice this goes down as *R* increases. But consider M_{21}: the little loop has a dipole-like field, certainly at large distances, like *R*. This field goes through the large loop.

But how can the amount of *this* field going through the big loop go *down* as the size of the big loop increases? (It must, since $M_{12} = M_{21}$, and M_{12} certainly goes down.)

Because if the big loop is infinitely large, the total flux through the big loop from a current in the little loop is zero! The magnetic flux goes up in the middle, through the little loop, then back down again outside the little loop (remember the magnetic field lines just circulate around). For a large but finite big loop, the net flux through the big loop from a current in the little loop is just the negative of the flux passing outside the big loop.

Self Inductance: Energy Stored in the Magnetic Field

We've already discussed back emf, a changing current in a coil generates an emf opposing the change in current. The ratio of the induced emf to the rate of change of current is called **self inductance**, and written *L*,

$$\mathcal{E} = -LdI / dt.$$

(As usual, the minus sign is there to remind us that the emf is opposing the current increase—it is necessary to use Lenz' law, or equivalently energy considerations, to find emf direction in any particular case.)

As increasing current is supplied to a coil, this back emf forces the external power supply to do work against it at a rate $P = I\mathcal{E} = LIdI/dt$.

Therefore the total work done is increasing the current from zero to *I* is $L\int IdI = \frac{1}{2}LI^2$.

This is directly analogous to charging a capacitor: in that case, it took more work to add more charge, and we saw the energy was stored in the electric field, with density $\frac{1}{2}\varepsilon_0 E^2$. For the inductance, the energy is similarly stored in the **magnetic** field. We can check that for a solenoid: for *N* turns and length ℓ, the field inside the solenoid $B = \mu_0 NI/\ell$.

The self inductance is the total flux through the *N* turns **for unit current**, take the cross-sectional area to be *A*, this is

$$L = NAB = \mu_0 N^2 A/\ell.$$

Writing $I = \ell B/\mu_0 N$, we find

$$\tfrac{1}{2}LI^2 = \frac{1}{2}\frac{\mu_0 N^2 A}{\ell}\left(\frac{\ell B}{\mu_0 N}\right)^2 = \tfrac{1}{2}A\ell B^2/\mu_0$$

That is, the magnetic energy density is $B^2/2\mu_0$. Recalling that $\mu_0 = 4\pi\times 10^{-7}$, a cubic meter at one tesla contains megajoules, highly relevant for designers of big particle accelerators, where the current is flowing in superconductor. If these are accidently heated, they become normal resistors, with consequent sudden—explosive—loss of field. This has happened.

We can use this energy density approach to find the inductance of a coaxial cable quite easily:

Between r_1, r_2 the magnetic field is $B = \mu_0 I/2\pi r$, so the energy density per unit length is

$$\frac{1}{2\mu_0}\int_{r_1}^{r_2} B^2 2\pi r dr = \frac{1}{2\mu_0}\int_{r_1}^{r_2}\left(\frac{\mu_0 I}{2\pi r}\right)^2 2\pi r dr = \frac{1}{2}\left(\frac{\mu_0}{2\pi}\right)\ln\frac{r_2}{r_1} I^2$$

from which $L = \frac{\mu_0}{2\pi}\ln\frac{r_2}{r_1}$ per meter. Notice that letting r_2 go to infinity, we conclude that a single infinitely long wire of radius r_1 has infinite magnetic energy density **per unit length**! Of course, there is no such thing as an infinitely long wire, and the divergence is very slow. It is also apparent that the self inductance of, for example, a single loop of wire must depend on the radius of the *wire* as well as that of the loop.