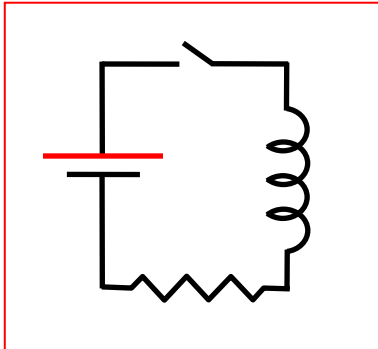# Physics 2415 Lecture 23: *LR*, *LC* and *LRC* Circuits

Michael Fowler, UVa, 10/25/09

LR Circuits

A battery is connected to *L*, *R* in series: $V_0 = IR + L\dfrac{dI}{dt}$.
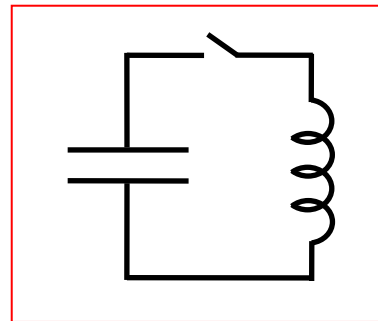
(*Quick sign check*: on first connecting the battery, *L* will oppose the current increasing from zero—so for positive dI/dt, the inductance is working against the battery.)

The math here is very similar to the capacitor:

$$\frac{dI}{V_0 - IR} = \frac{dt}{L}$$

The integration is routine, and $I = \dfrac{V_0}{R}\left(1 - e^{-t/\tau}\right)$ with $\tau = L/R$.

LC Circuits: Oscillations

We first assume there's zero resistance, just *C* and *L* in series, with an open switch. We charge *C*, then close the switch. Charge will begin to flow through the inductance from one plate of *C* to the other. The current $I = -dQ/dt$ will build up, but its **rate of increase will depend on the inductance**, the back emf LdI/dt balancing the voltage Q/C of the capacitor.

That is, $\dfrac{Q}{C} = L\dfrac{dI}{dt} = -L\dfrac{d^2Q}{dt^2}$ or $L\dfrac{d^2Q}{dt^2} = -\dfrac{Q}{C}$.

Now **compare this charge equation with the equation for displacement of a mass on a spring**: suppose the mass can slide on a smooth table, the spring being attached to a wall.   The equation is:

$$m\frac{d^2x}{dt^2} = -kx.$$

Here *m* is the mass, *k* the spring constant, and *x* the linear displacement from the spring's relaxed length position. It's clear that **these two equations are mathematically identical**!

Wall

Block on frictionless surface
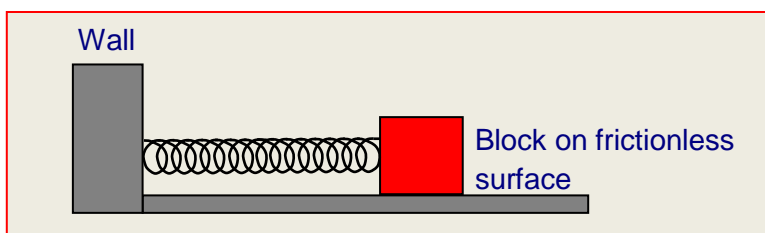
It's worth seeing what corresponds to what. The inductance *L* corresponds

to the mass m. These are both "inertial" terms: once the mass is moving, its mass or inertia keeps it going, so if it's pulled to one side then let go, the spring pulls it back, accelerating it, but even when the spring is back to its natural length the mass keeps moving and takes the mass the same distance the other way before stopping. For the *LC* circuit, the current is analogous to the velocity of the mass, and the inductance opposes change in current just as mass opposes change in velocity: when the initially charged capacitor has completely discharged, the inductance keeps the current going until the capacitor has its initial charge reversed.

We can see from the equations that the spring constant k corresponds to the inverse of the capacitance, $1/C$, so a bigger capacitance is like a softer spring. A larger capacitance can absorb charge more easily, just as a softer spring can be more easily stretched.

For the mass on the spring on a frictionless surface, pulled aside by x_0 and let go, $x = x_0 \cos \omega t$, here $\omega = \sqrt{k/m}$.

Similarly, if at $t = 0$ there is zero current and charge Q_0 on the capacitor, $Q = Q_0 \cos \omega t$, where $\omega = 1/\sqrt{LC}$.

The trading of **potential energy** stored in the spring and **kinetic energy** in the moving mass, with overall energy conserved at all times, is mirrored here in the trading **of electric field energy** in the capacitor and **magnetic field energy** in the inductance.

To check **conservation of energy** here, note that $Q = Q_0 \cos \omega t$ means that $I = -dQ/dt = Q_0 \omega \sin \omega t$, and total energy

$$U_{\text{total}} = \frac{Q^2}{2C} + \frac{LI^2}{2} = \frac{Q_0^2 \cos^2 \omega t}{2C} + \frac{LQ_0^2 \omega^2 \sin^2 \omega t}{2} = \frac{Q_0^2}{2C}$$

using $\omega^2 = 1/LC.$

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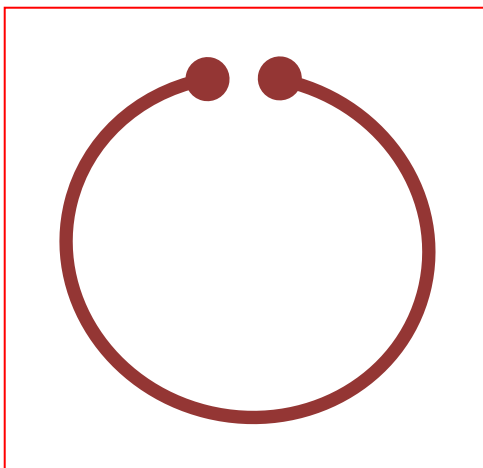
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It's also worth mentioning here that this *L, C* circuit can very simple: for example, a ring with a gap:

The two balls are the capacitor: begin with one of them positive, the other equally negative. The ring is of course inductor: a current going around it creates a magnetic field.

Initially, then, we have an electric dipole, with the corresponding field. But this is an oscillating field. The magnetic field from the current also oscillates, and we know that creates another electric field. The picture is not yet complete—there is one crucial element

missing—but we can begin to see how an oscillating circuit might emit electromagnetic waves.

LRC Circuit

How does this picture change if we include resistance? Obviously, the same way, essentially, the oscillating mass on a spring changes if we include air or fluid resistance: the oscillations are damped.

More formally, we must add a term *IR* to the voltage sum:

$$L\frac{d^2Q}{dt^2} + R\frac{dQ}{dt} + \frac{Q}{C} = 0$$
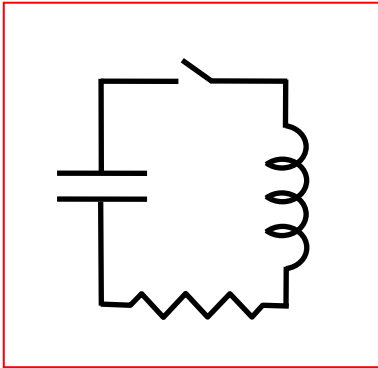
This equation for *Q* is **identical** to the equation of motion for *x* for a **damped** mass on a spring: imagine a mass in molasses.   *L* is the mass, 1/*C* is the spring constant *k* and *R* is the damping term *b*: see that earlier work.

This means that the behavior of *Q* as a function of time is exactly the same as the behavior of *x* in the damped oscillator. In particular, there are two different regimes: lightly damped, where the sign of charge on the capacitor oscillates back and forth, gradually dying away, and heavily damped, where the charge slowly drains through the system, decreasing exponentially, with no oscillation. The boundary between the two is critical damping, and occurs when the damping term $R^2 = 4L/C$.

It also follows from the exact analogy with the damped mechanical oscillator that introducing damping (resistance) lowers the frequency of the oscillations, but for small damping this is a very small effect. However, as the damping approaches the critical value $R^2 = 4L/C$, the frequency goes to zero.