20. Parametric Resonance

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Introduction

(*Following Landau para 27*)

A one-dimensional simple harmonic oscillator, a mass on a spring,

$$\frac{d}{dt}(m\dot{x}) + kx = 0$$

has two parameters, m and k. For some systems, the parameters can be changed externally (an example being the length of a pendulum if at the top end the string goes over a pulley).

We are interested here in the system's response to some externally imposed periodic variation of its parameters, and in particular we'll be looking at *resonant* response, meaning large response to a small imposed variation.

Note first that imposed variation in the mass term is easily dealt with, by simply redefining the time variable to $d\tau = dt/m(t)$, meaning $\tau = \int \frac{dt}{m(t)}$. Then

$$\frac{d}{dt}\left(m\frac{dx}{dt}\right) = \frac{1}{m}\frac{d}{d\tau}\left(m\frac{1}{m}\frac{dx}{d\tau}\right) = \frac{1}{m}\frac{d^2x}{d\tau^2},$$

and the equation of motion becomes $\frac{d^2x}{d\tau^2} + m(\tau)kx = 0.$

This means we can always transform the equation so all the parametric variation is in the spring constant, so we'll just analyze the equation

$$\frac{d^2x}{dt^2} + \omega^2(t)x = 0.$$

Furthermore, since we're looking for *resonance* phenomena, we will only consider a small parametric variation at a single frequency, that is, we'll take

$$\omega^2(t) = \omega_0^2(1 + h\cos\Omega t),$$

where $h \ll 1$, and h is positive (a trivial requirement—just setting the time origin).

(*Note*: We prefer Ω where Landau uses γ, which is often used for a resonance *width* these days.)

We have now a driven oscillator:

$$\frac{d^2x}{dt^2}+\omega_0^2 x=-\omega_0^2 xh\cos\Omega t.$$

How does this differ from our previous analysis of a driven oscillator? In a very important way!

The amplitude x is a factor in the driving force.

For one thing, this means that if the oscillator is initially at rest, it stays that way, in contrast to an ordinary externally driven oscillator. But if the amplitude increases, so does the driving force. This can lead to an *exponential* increase in amplitude, unlike the linear increase we found earlier with an external driver. (Of course, in a real system, friction and nonlinear potential terms will limit the growth.)

What frequencies will prove important in driving the oscillator to large amplitude? It responds best, of course, to its natural frequency ω_0. But if it is in fact already oscillating at that frequency, then the driving force, *including the factor of* x, is proportional to

$$\cos\omega_0 t\cos\Omega t=\tfrac{1}{2}\cos(\Omega-\omega_0)t+\tfrac{1}{2}\cos(\Omega+\omega_0)t,$$

with no component at the natural frequency ω_0 for a general Ω.

The simplest way to get resonance is to take $\Omega=2\omega_0$. Can we understand this physically? Yes. Imagine a mass oscillating backwards and forwards on a spring, and the spring force increases just after those points where the mass is furthest away from equilibrium, so it gets an extra tug inwards twice a cycle. This will feed in energy. (You can drive a swing this way.) In contrast, if you drive at the natural frequency, giving little push inwards just after it begins to swing inwards from one side, then you'll be giving it a little push *outwards* just after it begins to swing back from the other side. Of course, if you push only from one side, like swinging a swing, this works—but it isn't a single frequency force, the next harmonic is doing most of the work.

Resonance near Double the Natural Frequency

From the above argument, the place to look for resonance is close to $\Omega=2\omega_0$. Landau takes

$$\ddot{x}+\omega_0^2\left[1+h\cos(2\omega_0+\varepsilon)t\right]x=0$$

and, bearing in mind that we're looking for oscillations close to the natural frequency, puts in

$$x=a(t)\cos(\omega_0+\tfrac{1}{2}\varepsilon)t+b(t)\sin(\omega_0+\tfrac{1}{2}\varepsilon)t,$$

with $a(t), b(t)$ slowly varying.

It's important to realize that this is an *approximate* approach. It neglects nonresonant frequencies which must be present in small amounts, for example

$$\cos\left(\omega_0+\tfrac{1}{2}\varepsilon\right)t\cos\left(2\omega_0+\varepsilon\right)t=\tfrac{1}{2}\cos 3\left(\omega_0+\tfrac{1}{2}\varepsilon\right)t+\tfrac{1}{2}\cos\left(\omega_0+\tfrac{1}{2}\varepsilon\right)t$$

and the $3\left(\omega_0+\tfrac{1}{2}\varepsilon\right)$ term is thrown away.

And, since the assumption is that $a(t), b(t)$ are slowly varying, their second derivatives are dropped too, leaving just

$$\begin{aligned}\ddot{x}=&-2\dot{a}(t)\omega_0\sin\left(\omega_0+\tfrac{1}{2}\varepsilon\right)t-a(t)\left(\omega_0^2+\omega_0\varepsilon\right)\cos\left(\omega_0+\tfrac{1}{2}\varepsilon\right)t\\&+2\dot{b}(t)\omega_0\cos\left(\omega_0+\tfrac{1}{2}\varepsilon\right)t-b(t)\left(\omega_0^2+\omega_0\varepsilon\right)\sin\left(\omega_0+\tfrac{1}{2}\varepsilon\right)t.\end{aligned}$$

This must equal

$$-\omega_0^2\left[1+h\cos\left(2\omega_0+\varepsilon\right)t\right]\left[a(t)\cos\left(\omega_0+\tfrac{1}{2}\varepsilon\right)t+b(t)\sin\left(\omega_0+\tfrac{1}{2}\varepsilon\right)t\right].$$

Keeping only the resonant terms, we take $\cos\left(\omega_0+\tfrac{1}{2}\varepsilon\right)t\cdot\cos\left(2\omega_0+\varepsilon\right)t=\tfrac{1}{2}\cos\left(\omega_0+\tfrac{1}{2}\varepsilon\right)t$ and $\sin\left(\omega_0+\tfrac{1}{2}\varepsilon\right)t\cdot\cos\left(2\omega_0+\varepsilon\right)t=-\tfrac{1}{2}\sin\left(\omega_0+\tfrac{1}{2}\varepsilon\right)t$, so this expression becomes

$$\begin{aligned}&-\omega_0^2\left[1+h\cos\left(2\omega_0+\varepsilon\right)t\right]\left[a(t)\cos\left(\omega_0+\tfrac{1}{2}\varepsilon\right)t+b(t)\sin\left(\omega_0+\tfrac{1}{2}\varepsilon\right)t\right]\\&=-\omega_0^2\left[a(t)\cos\left(\omega_0+\tfrac{1}{2}\varepsilon\right)t+b(t)\sin\left(\omega_0+\tfrac{1}{2}\varepsilon\right)t+\tfrac{1}{2}ha(t)\cos\left(\omega_0+\tfrac{1}{2}\varepsilon\right)t-\tfrac{1}{2}hb(t)\sin\left(\omega_0+\tfrac{1}{2}\varepsilon\right)t\right]\end{aligned}$$

The equation becomes:

$$\begin{aligned}\ddot{x}=&-2\dot{a}(t)\omega_0\sin\left(\omega_0+\tfrac{1}{2}\varepsilon\right)t-a(t)\left(\omega_0^2+\omega_0\varepsilon\right)\cos\left(\omega_0+\tfrac{1}{2}\varepsilon\right)t\\&+2\dot{b}(t)\omega_0\cos\left(\omega_0+\tfrac{1}{2}\varepsilon\right)t-b(t)\left(\omega_0^2+\omega_0\varepsilon\right)\sin\left(\omega_0+\tfrac{1}{2}\varepsilon\right)t\\==&-\omega_0^2\left[a(t)\cos\left(\omega_0+\tfrac{1}{2}\varepsilon\right)t+b(t)\sin\left(\omega_0+\tfrac{1}{2}\varepsilon\right)t+\tfrac{1}{2}ha(t)\cos\left(\omega_0+\tfrac{1}{2}\varepsilon\right)t-\tfrac{1}{2}hb(t)\sin\left(\omega_0+\tfrac{1}{2}\varepsilon\right)t\right]\end{aligned}$$

The zeroth-order terms cancel between the two sides, leaving

$$\begin{aligned}&-2\dot{a}(t)\omega_0\sin\left(\omega_0+\tfrac{1}{2}\varepsilon\right)t-a(t)\omega_0\varepsilon\cos\left(\omega_0+\tfrac{1}{2}\varepsilon\right)t+2\dot{b}(t)\omega_0\cos\left(\omega_0+\tfrac{1}{2}\varepsilon\right)t-b(t)\omega_0\varepsilon\sin\left(\omega_0+\tfrac{1}{2}\varepsilon\right)t\\&=-\omega_0^2\left[\tfrac{1}{2}ha(t)\cos\left(\omega_0+\tfrac{1}{2}\varepsilon\right)t-\tfrac{1}{2}hb(t)\sin\left(\omega_0+\tfrac{1}{2}\varepsilon\right)t\right]\end{aligned}$$

Collecting the terms in $\sin\left(\omega_0+\tfrac{1}{2}\varepsilon\right)t,\ \cos\left(\omega_0+\tfrac{1}{2}\varepsilon\right)t$:

$$-\left(2\dot{a}+b\varepsilon+\tfrac{1}{2}h\omega_0 b\right)\omega_0\sin\left(\omega_0+\tfrac{1}{2}\varepsilon\right)t+\left(2\dot{b}(t)-a\varepsilon+\tfrac{1}{2}h\omega_0 a\right)\omega_0\cos\left(\omega_0+\tfrac{1}{2}\varepsilon\right)t=0.$$

The sine and cosine can't cancel each other, so the two coefficients must both be identically zero. This gives two first order differential equations for the functions $a(t), b(t)$, and we look for exponentially increasing functions, proportional to $a(t) = ae^{st}$, $b(t) = be^{st}$, which will be solutions provided

$$\begin{aligned} sa + \tfrac{1}{2}\left(\varepsilon + \tfrac{1}{2}h\omega_0\right)b = 0, \\ \tfrac{1}{2}\left(\varepsilon - \tfrac{1}{2}h\omega_0\right)a - sb = 0. \end{aligned}$$

The amplitude growth rate is therefore

$$s^2 = \tfrac{1}{4}\left[\left(\tfrac{1}{2}h\omega_0\right)^2 - \varepsilon^2\right].$$

Parametric resonance will take place if s is real, that is, if

$$-\tfrac{1}{2}h\omega_0 < \varepsilon < \tfrac{1}{2}h\omega_0,$$

a band of width $h\omega_0$ about $2\omega_0$.

Example: Pendulum Driven at near Double the Natural Frequency

A simple pendulum of length ℓ, mass m is attached to a point which oscillates vertically $y = a\cos\Omega t$. Measuring y downwards, the pendulum position is

$$x = \ell\sin\phi,\ y = a\cos\Omega t + \ell\cos\phi.$$

The Lagrangian

$$\begin{aligned} L &= \tfrac{1}{2}m\left(\dot{x}^2 + \dot{y}^2\right) + mg\ell\cos\phi \\ &= \tfrac{1}{2}m\left(\ell^2\cos^2\phi\right)\dot{\phi}^2 + \tfrac{1}{2}m\left(a\Omega\sin\Omega t + \ell\dot{\phi}\sin\phi\right)^2 + mg\ell\cos\phi \\ &= \tfrac{1}{2}m\ell^2\dot{\phi}^2 - ma\ell\Omega\sin\Omega t\frac{d}{dt}\cos\phi + a^2\Omega^2\sin^2\Omega t + mg\ell\cos\phi \end{aligned}$$

The purely time-dependent term will not affect the equations of motion, so we drop it, and since the equations are not affected by adding a total derivative to the Lagrangian, we can integrate the second term by parts (meaning we're dropping a term $\frac{d}{dt}\left(ma\ell\Omega\sin\Omega t\cos\phi\right)$) to get

$$L = \tfrac{1}{2}m\ell^2\dot{\phi}^2 + ma\ell\Omega^2\cos\Omega t\cos\phi + mg\ell\cos\phi.$$

(We've also dropped the term $mga\cos\Omega t$ from the potential energy term—it has no ϕ or $\dot{\phi}$ dependence, so will not affect the equations of motion.)

The equation for small oscillations is

$$\ddot{\phi} + \omega_0^2 \left[1 + \left(4a/\ell\right)\cos\left(2\omega_0 + \varepsilon\right)t\right]\phi = 0, \quad \omega_0^2 = g/\ell.$$

Comparing this with

$$\ddot{x} + \omega_0^2 \left[1 + h\cos\left(2\omega_0 + \varepsilon\right)t\right]x = 0$$

we see that $4a/\ell \equiv h$, so the parametric resonance range around $2\omega_0 = 2\sqrt{g/\ell}$ is of width $\frac{1}{2}h\omega_0 = 2a\sqrt{g/\ell^3}$.