9.3 Magnetic Dipole and Electric Quadrupole.

9.3.1

a. We start again from the exact formula

$$\mathbf{A}(\mathbf{x}) = \frac{1}{c}\int \mathbf{J}(\mathbf{x}')\,\frac{e^{ik|\mathbf{x}-\mathbf{x}'|}}{|\mathbf{x}-\mathbf{x}'|}d^3x',$$

insert

$$\frac{e^{ik|\mathbf{x}-\mathbf{x}'|}}{|\mathbf{x}-\mathbf{x}'|} \simeq e^{ikr}e^{-ik\mathbf{n}\cdot\mathbf{x}'}\left(\frac{1}{r}+\frac{\mathbf{n}\cdot\mathbf{x}'}{r^2}\right)$$

from 9.1.3.c, expand

$$e^{-ik\mathbf{n}\cdot\mathbf{x}'} \simeq 1 - ik\mathbf{n}\cdot\mathbf{x}'$$

and pick up the terms proportional to $\mathbf{n}\cdot\mathbf{x}'$:

$$\mathbf{A}(\mathbf{x}) = \frac{e^{ikr}}{cr}\left(\frac{1}{r}-ik\right)\int \mathbf{J}(\mathbf{x}')\mathbf{n}\cdot\mathbf{x}'d^3x'. \tag{3.1}$$

We recall that this is valid for $kd \ll 1$ and leads to an expansion in powers of d/r for $kr < 1$ and in powers of kd for large $kr \gg 1$.

b. The integrand in (3.1) involves the tensor $J_\alpha x'_\beta$. The symmetric and antisymmetric parts of this tensor lead respectively to the electric quadrupole and magnetic dipole fields.

c. The antisymmetric part,

$$(1/2)(J_\alpha x'_\beta - x'_\alpha J_\beta)$$

gives

$$\frac{1}{2}\left[\mathbf{J}(\mathbf{n}\cdot\mathbf{x}') - \mathbf{x}'(\mathbf{n}\cdot\mathbf{J})\right],$$

which is recognized as $-(1/2)\mathbf{n}\times(\mathbf{x}'\times\mathbf{J})$. Hence the magnetic dipole vector potential is

$$\mathbf{A}(\mathbf{x}) = ik(\mathbf{n}\times\mathbf{m})\frac{e^{ikr}}{r}\left(1-\frac{1}{ikr}\right)$$

where

$$\mathbf{m} = \frac{1}{2c}\int \mathbf{x}'\times\mathbf{J}(\mathbf{x}')d^3x'$$

is the magnetic moment.

9.3.2

a. The task of finding $\mathbf{E}$ and $\mathbf{B}$ for the *magnetic dipole* is greatly simplified if one recalls that only the divergenceless part of the current contributes to the magnetization (see **c.**). Hence the charge density ρ vanishes by the continuity equation, the scalar potential ϕ vanishes and $\mathbf{E} = -(1/c)\partial \mathbf{A}/\partial t = i(\omega/c)\mathbf{A} = ik\mathbf{A}$. Thus

$$\mathbf{E} = -k^2(\mathbf{n} \times \mathbf{m})\frac{e^{ikr}}{r}\left(1 - \frac{1}{ikr}\right) \tag{3.2}$$

$$\mathbf{B} = \boldsymbol{\nabla} \times \mathbf{A} = k^2\mathbf{m}_\perp\frac{e^{ikr}}{r} + \boldsymbol{\nabla}\left[\mathbf{m} \cdot \boldsymbol{\nabla}\left(\frac{1}{r}\right)\right] e^{ikr}(1 - ikr). \tag{3.3}$$

The calculation leading to $\mathbf{B}$ is exactly the same as the calculation giving $\mathbf{E}$ for the electric dipole.

b. The analogy between magnetic and electric dipole is another example of the symmetry of Maxwell's equations *outside the sources* in the electric and magnetic quantities. Once the electric problem is solved, we need only let $\mathbf{E} \to \mathbf{H}$, $\mathbf{B} \to -\mathbf{D}$, $\varepsilon \to \mu$ and $\mathbf{p} \to \mathbf{m}$.

c. Jackson shows on page 188 that a magnetization density $\mathbf{M}$ has the same effect as a current $\mathbf{J} = c\boldsymbol{\nabla} \times \mathbf{M}$, which is clearly divergenceless. If further proof is needed, use

$$\frac{1}{2}\varepsilon_{\alpha\beta\gamma}\int \frac{\partial}{\partial x_\beta}(r^2 J_\gamma)d^3x = \frac{1}{2}\int r^2\varepsilon_{\alpha\beta\gamma}\frac{\partial J_\gamma}{\partial \beta}d^3x + \int \varepsilon_{\alpha\beta\gamma}x_\beta J_\gamma d^3x.$$

The l.h.s. gives a vanishing integral over the boundary, so that

$$\mathbf{m} = -\frac{1}{2}\int r^2\boldsymbol{\nabla} \times \mathbf{J}(\mathbf{x})d^3\mathbf{x}. \tag{3.4}$$

Now $\mathbf{J}$ can be split, $\mathbf{J} = \mathbf{J}_l + \mathbf{J}_t$ with $\boldsymbol{\nabla} \times \mathbf{J}_l = 0$ and $\boldsymbol{\nabla} \cdot \mathbf{J}_t = 0$ and clearly only the divergenceless part $\mathbf{J}_t$ contributes to $\mathbf{m}$.

9.3.3

a. For the symmetric part of $J_\alpha x'_\beta$ we use

$$\sum_\gamma \frac{\partial}{\partial x'_\gamma}(x'_\alpha x'_\beta J_\alpha) = J_\alpha x'_\beta + J_\beta x'_\alpha + x'_\alpha x'_\beta \sum_\gamma \frac{\partial J_\gamma}{\partial x'_\gamma},$$

which gives, integrating and discarding the boundary term:

$$\frac{1}{2}\int \left[(\mathbf{n}\cdot\mathbf{x}')\,\mathbf{J} + (\mathbf{n}\cdot\mathbf{J})\,\mathbf{x}'\right] d^3x' = -\frac{1}{2}\int \mathbf{x}'\,(\mathbf{n}\cdot\mathbf{x}')\,\boldsymbol{\nabla}\cdot\mathbf{J}(\mathbf{x}')d^3x'.$$

b. Note that only $\mathbf{J}_l$ (see previous page) contributes now. The same was true for the electric dipole part. Using the continuity equation, $\boldsymbol{\nabla}\cdot\mathbf{J} = i\omega\rho$, we find from (3.1) that the symmetric part of $J_\alpha x'_\beta$ gives rise to the vector potential

$$\mathbf{A}(\mathbf{x}) = -\frac{k^2}{2}\frac{e^{ikr}}{r}\left(1 - \frac{1}{ikr}\right)\int \mathbf{x}'\,(\mathbf{n}\cdot\mathbf{x}')\,\rho(\mathbf{x}')d^3x'. \qquad (3.5)$$

c. This part of $\mathbf{A}$ is now seen to be given by the tensor $x'_\alpha x'_\beta \rho(\mathbf{x}')$ of the second moments of the charge distribution. Only the traceless part of this tensor actually contributes to the fields for $\omega \neq 0$. This follows from the fact that $r'^2\delta_{\alpha\beta}\,\rho$ leads to

$$\mathbf{A} = \frac{ik}{2}\left(\int r'^2\rho(\mathbf{x}')d^3x'\right)\boldsymbol{\nabla}\left(\frac{e^{ikr}}{r}\right),$$

which implies $\boldsymbol{\nabla}\times\mathbf{A} = \mathbf{0}$.

d. We therefore define the (electric) *quadrupole moment tensor*

$$Q_{\alpha\beta} = \int \left(3x_\alpha x_\beta - r^2\delta_{\alpha\beta}\right)\rho(\mathbf{x})d^3x \qquad (3.6)$$

so that $\sum_\alpha Q_{\alpha\alpha} = 0$. When the tensor is referred to principal axes, only two numbers (say Q_{33} and $(1/2)(Q_{11} - Q_{22})$) specify the properties of the system.

e. We also define a vector $\mathbf{Q}(\mathbf{n})$ by

$$Q_\alpha = \sum_\beta Q_{\alpha\beta} n_\beta. \qquad (3.7)$$

9.3.4

a. The quadrupole fields are obtained from (3.5) by $\mathbf{B} = \boldsymbol{\nabla} \times \mathbf{A}$, $\mathbf{E} = (i/k)\boldsymbol{\nabla} \times \mathbf{B}$. In the radiation zone, $kr \gg 1$, we can replace $\boldsymbol{\nabla}\times$ by $ik\mathbf{n}\times$, because the derivatives of $\exp(ikr)$ give the leading order. Therefore, with the definition (3.7):

$$\mathbf{B} = -\frac{ik^3}{6}\frac{e^{ikr}}{r}\mathbf{n} \times \mathbf{Q}(\mathbf{n}) \qquad\qquad \mathbf{E} = -\mathbf{n} \times \mathbf{B} \tag{3.8}$$

$$\frac{dP}{d\Omega} = \frac{cr^2}{8\pi}\,\mathrm{Re}\left[\mathbf{n} \cdot \mathbf{E} \times \mathbf{B}^*\right] = \frac{ck^6}{288\pi}\left|Q_\perp(\mathbf{n})\right|^2 \tag{3.9}$$

with

$$Q_\perp(\mathbf{n}) = [\mathbf{n} \times \mathbf{Q}(\mathbf{n})] \times \mathbf{n} = \mathbf{Q}(\mathbf{n}) - \mathbf{n}\left[\mathbf{Q}(\mathbf{n}) \cdot \mathbf{n}\right]$$

so that

$$|Q_\perp|^2 = \mathbf{Q}^*\cdot\mathbf{Q} - |\mathbf{n} \cdot \mathbf{Q}|^2 = \sum_{\alpha\beta\gamma} Q^*_{\alpha\beta} Q_{\alpha\gamma} n_\beta n_\gamma - \sum_{\alpha\beta\gamma\delta} Q^*_{\alpha\beta} Q_{\gamma\delta} n_\alpha n_\beta n_\gamma n_\delta.$$

b. To find the total radiated power we need the integrals

$$\int n_\beta n_\gamma d\Omega = \frac{4\pi}{3}\delta_{\beta\gamma}$$

$$\int n_\alpha n_\beta n_\gamma n_\delta d\Omega = \frac{4\pi}{15}\left[\delta_{\alpha\beta}\delta_{\gamma\delta} + \delta_{\alpha\gamma}\delta_{\beta\delta} + \delta_{\alpha\delta}\delta_{\beta\gamma}\right].$$

It is clear, for instance, that

$$\int n_1 n_2 d\Omega = \int xy/r^2 d\Omega$$

vanishes because the integrand is odd in x. One is then reduced to compute

$$\int n_3^2 d\Omega = 2\pi \int \cos^2\theta d(\cos\theta) = 4\pi/3, \int n_3^4 d\Omega = 2\pi \int \cos^4\theta d(\cos\theta) = 4\pi/15$$

and

$$\int n_3^2 n_1^2 d\Omega = \int \cos^2\theta \sin^2\theta d(\cos\theta) \int \cos^2\varphi d\varphi = 4\pi/15.$$

c. We find then, using also $\sum_\alpha Q_{\alpha\alpha} = 0$,

$$P = \int \frac{dP}{d\Omega} d\Omega = \frac{ck^6}{360}\sum_{\alpha\beta}\left|Q_{\alpha\beta}\right|^2.$$

9.3.5

a. In practice it is convenient to refer $Q_{\alpha\beta}$ to its principal axes, which are often obvious from the symmetry of the system. For instance, if the system has an n-fold rotation axis, with $n \geq 3$, this axis is a principal axis of $Q_{\alpha\beta}$, say x_3. Further, in this case $Q_{11} = Q_{22}$ and both equal $-Q_{33}/2$, since $Q_{11} + Q_{22} + Q_{33}$ must vanish. Then Q_{33} is simply called the quadrupole moment.

b. With a diagonal $Q_{\alpha\beta}$ we find from 4.a

$$|\mathbf{Q}_{\perp}(\mathbf{n})|^2 = \sum_{\alpha} |Q_{\alpha\alpha}|^2 n_{\alpha}^2 - \sum_{\alpha\gamma} Q_{\alpha\alpha}^* n_{\alpha}^2 Q_{\gamma\gamma} n_{\gamma}^2.$$

c. If $Q_{11} = Q_{22} = -Q_{33}/2$ and $|Q_{33}| = Q_0$

$$|Q_{\perp}(\mathbf{n})|^2 = Q_0^2 \left(\frac{\sin^2\theta}{4} + \cos^2\theta \right) - Q_0^2 \left(-\frac{\sin^2\theta}{2} + \cos^2\theta \right)^2 = \frac{9}{4} Q_0^2 \sin^2\theta \cos^2\theta$$

so that

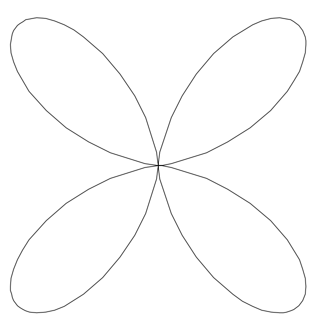
$$\frac{dP}{d\Omega} = \frac{ck^6}{128\pi} Q_0^2 \sin^2\theta \cos^2\theta.$$

d. The total radiated power is now

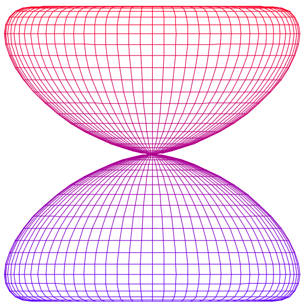
$$P = \frac{ck^6 Q_0^2}{240}.$$

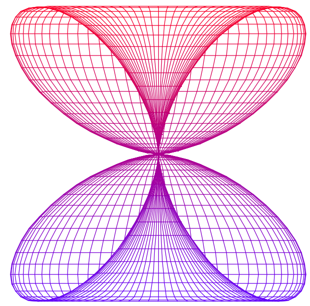
e. The radiation pattern is obtained by rotating the "cloverleaf" in the picture on this page around the vertical axis (the z axis). Three-dimensional views are on next page. The intercept in any direction is proportional to the intensity radiated in that direction.

Here is a picture of $\sin^2\theta \cos^2\theta$ in cross-section



Here is a 3-d view of $\sin^2\theta\cos^2\theta$ from the side, and a cut of the same





Here are full views at 18 and 36 degrees from the vertical (the z axis), and a cut view at 18 degrees: