5. Electrostatics III: Capacitances and the Reciprocation Theorem

Electrostatics of Systems of Conductors

We know by now that in electrostatic equilibrium, the charges are at rest (by definition of electro*static*!), so, since they are free to move in a conductor, the electric field must be zero inside the conductor, and in fact perpendicular to the surface going outward from the conductor into an insulator or vacuum. Therefore, the potential must be constant throughout the conductor.

Capacitance

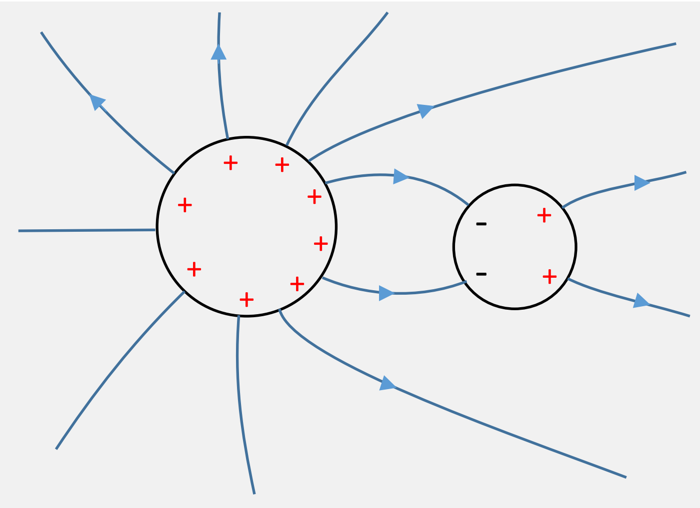
Let's start with *self*-capacitance, think of a single isolated conductor carrying a charge Q. In electrostatic equilibrium the entire conductor will be at the same potential, $\varphi(\vec{r}) = V$, say, for any $\vec{r}$ in or on the body. (Sorry we also use V for volume, but this is standard.)

The self-capacitance (or just capacitance) is defined as

$$C = Q / V.$$

It can be understood as a measure of how much charge can be loaded on the conductor before it reaches a given voltage (potential). Experimentally, the relationship is linear for a rigid conductor over a wide range. The simplest example is the capacitance of an isolated spherical conductor of radius R. If the sphere has charge Q, its surface is at potential $\varphi = Q / 4\pi\varepsilon_0 R$, so its capacitance $C = 4\pi\varepsilon_0 R$.

Grounding a conductor means connecting it via a wire, say, to Earth, so that (in electrostatic equilibrium) it will be at the same potential as the Earth, which we take as our zero. (Reasonably assuming that the Earth itself has no net charge.)



Charge on conductor C_1 raises potential of uncharged C_2

Now imagine we have two conductors, take two spheres, say, both grounded so they are at zero potential. Now remove the connections to Earth, and put a charge Q on one of them only. What, if anything, happens to the potential of the other sphere? It's helpful to do a qualitative sketch of the electric field lines. The charge on the first sphere will cause some redistribution of charge on the second, so some field lines from the first will end on the second. Furthermore, some field lines will begin on the far side of the second sphere and go to infinity. It is then clear that in fact the second sphere is now at a *nonzero* potential, it takes work to come in along that field line from infinity. And, as is evident from the linearity, the electrostatic potential, the potential of the second sphere depends *linearly* on the charge of the first sphere.

The Capacitance Matrix

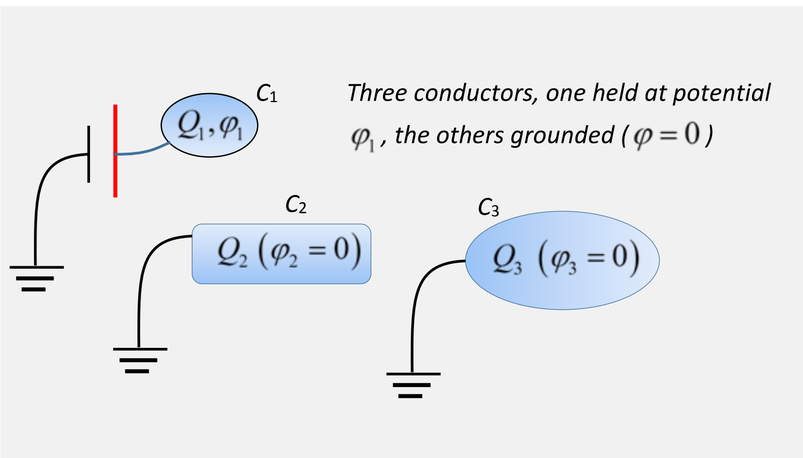
We've defined capacitance for a single conductor as $Q = CV = C\varphi$. I put that CV term in because you must have seen it written that way, using the term voltage rather than potential, in earlier courses. However, we're going to use φ's for potentials, not least because we want to use V for volume.

We'll now consider a system of n separate conductors, having charges Q_i and potentials φ_i. It's clear that there are linear connections between the potential on one conductor and the charges on the others, that is, there is a *capacitance matrix*:

$$Q_i = \sum_{j=1}^{n} C_{ij}\varphi_j.$$

To gain more insight into this expression, consider the following.

Take conductor #1 and use some voltage source to raise its potential to φ_1, at the same time have all the other conductors connected to ground, so all the other potentials are $\varphi_i = 0,\ i = 2,\ldots n$. Of course, this means that the charges on these other conductors will in general be nonzero, the presence of the charge Q_1 necessary to raise conductor #1 to φ_1 will attract charges from Earth to the other conductors. In terms of the capacitance matrix, for this configuration the i^{th} conductor has charge



$$Q_i^{(1)} = C_{i1}\varphi_1.$$

When the charges have come from Earth to the grounded conductors, we can then cut the wires, this will change nothing, as no current was flowing.

We then repeat with all but conductor #2 grounded, #2 being held at φ_2, and the i^{th} conductor in this scenario has charge

$$Q_i^{(2)} = C_{i2}\varphi_2.$$

Repeating the same procedure for all the conductors, each equation is legitimate, and, again invoking linearity, the *sums* of these solutions (summing both charges and potentials) are also electrostatically valid, therefore unique.

The bottom line is that, given any set of *potentials* on the family of conductors, this capacitance matrix automatically gives us the *charge* on each conductor. The catch, of course, is in actually evaluating the C_{ij}.

Note that the capacitance matrix element C_{11} is *not* the self-capacitance of the first conductor if it were in isolation: the charge coming from the ground to the other conductors affects the potential on the first conductor.

The Capacitance Matrix is Symmetric

The off-diagonal terms satisfy:

$$C_{ij} = C_{ji}.$$

That is, no matter how weirdly shaped two conductors A and B are, if putting a charge Q on A raises the potential of B by V, then putting the same magnitude charge Q on B raises the potential of A by the *same* V —certainly not obvious!

This is simple to prove using Green's theorem,

$$\int_V \left(\varphi \nabla^2 \psi - \psi \nabla^2 \varphi\right) dV = \int_S \left(\varphi \frac{\partial \psi}{\partial n} - \psi \frac{\partial \varphi}{\partial n}\right) da.$$

Suppose $\varphi(\vec{r})$ is the potential throughout space resulting from putting charge Q on A, and in particular φ_B is the potential of conductor B. Similarly, let $\psi(\vec{r})$ denote the potential from putting charge Q on B, in *a separate experiment*.

Now look at the equation: the volume integral on the left-hand side is zero, and in the surface integral the potentials are constant over the surfaces, these are conductors, so from Gauss' law, both term are of the form potential × charge on conductor. For the right hand side of the equation to match the left, go to zero, we must have

$$\varphi_B Q_A = \psi_A Q_B,$$

and we took $Q_A = Q_B = Q.$
Therefore

$$\psi_A = \varphi_B.$$

Green's Reciprocation Theorem (aka Reciprocity Relation)

The above proof of symmetry of the mutual capacitance coefficients is actually a special case of Green's reciprocation theorem:
Suppose in a bounded space a volume charge density ρ_A and boundary surface densities σ_A give an electrostatic potential φ_A. Now suppose that in that same space different charge densities, ρ_B, σ_B give potential φ_B.
Then

$$\int_V \rho_A \varphi_B dV + \int_S \sigma_A \varphi_B da = \int_V \rho_B \varphi_A dV + \int_S \sigma_B \varphi_A da.$$

The proof is simple: we'll just ignore the surface terms, they follow the same pattern.

Putting $\varphi_B(\vec{r}) = \dfrac{1}{4\pi\varepsilon_0} \displaystyle\int_V \frac{\rho_B(\vec{r}')}{|\vec{r} - \vec{r}'|}$ into the first term above,

$$\int_V \rho_A(\vec{r})\varphi_B(\vec{r})d^3r = \frac{1}{4\pi\varepsilon_0}\int_V \frac{\rho_A(\vec{r})\rho_B(\vec{r}')}{|\vec{r}-\vec{r}'|}d^3rd^3r'$$

which is clearly A, B symmetric.