10/17/17

2D Electrostatics

Why Bother with Two-Dimensional Potential Problems?

In fact, essentially two-dimensional problems arise quite a lot: fields inside conducting tubes and ducts, such as coaxial cables, long focusing magnets in particle beam physics, etc. Furthermore, quasi-2D systems are now common in electrical circuit design—and we'll give a nice example of how the techniques developed in this lecture are used to measure resistivity of thin samples of conducting material.

So we're dealing with the two-dimensional Laplace equation

$$\nabla^2 \varphi = \frac{\partial^2 \varphi}{\partial x^2} + \frac{\partial^2 \varphi}{\partial y^2} = 0.$$

A useful mental picture of possible solutions of this equation in an area, with specified boundary values along the edge, is provided by visualizing a rubber sheet, almost flat, covering the region, the sheet edge held at varying heights above a flat plane, edge height corresponding to the value of φ at that point. (No gravity here, by the way.)

The sheet height $h(x, y)$ will satisfy the Laplace equation—in fact, it's just the local static equation for a little square of rubber in equilibrium under the tension forces tugging on its four edges.

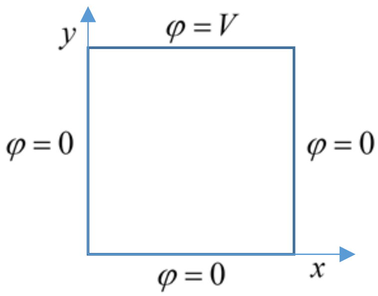
By the way, thinking about this rubber sheet, it's intuitively obvious that the potential cannot have a maximum or minimum away from the edges.

How to Solve 2D Electrostatic Problems

In this course, we're focusing on problems that can be solved with the standard mathematical tools: orthogonal functions (such as Fourier series), Green's functions, and complex variable methods. We won't cover numerical methods. We have to admit that this greatly restricts the two-dimensional geometries we can deal with in a practical way: mainly rectangles and circles, but with different parts of the curve held at different potentials. More difficult shapes are not difficult to solve computationally, the computing equivalent of the rubber sheet is to draw a square grid, put in plausible values at all the points, then repeatedly replace each point value by the average value of its four neighbors.

Orthogonal Functions

To illustrate the orthogonal functions approach, consider the problem of a square, one side held at potential V, the other three sides grounded.



For a function defined on a line, we can always use a Fourier series, the orthogonal functions being the sine waves if we have the function zero at the two ends. But we can't use sine waves in both directions: a product of two sines can't satisfy $\nabla^2\varphi = 0$. Or, think of the rubber sheet, you *could* use sine waves for a *vibrating* drumhead, but not for a static problem, as we have here.

For sides of unit length, it's clear that the functions $\varphi_n(x,y) = A_n \sin n\pi x \sinh n\pi y$ do satisfy $\nabla^2\varphi = 0$, and are zero on the appropriate three sides. We can then find the A_n by evaluating at the other side, $y = 1$. The result is $A_n = 0$ for even n, and $A_n = 2V_o / n\pi \sinh n\pi$ for odd n.

Of course, this Fourier method would also work for a more general $V(x)$ along the top, then by superposition of four such solutions, for any potential function on the four sides.

Complex Variable Method

We've shown that if a function of a complex variable is analytic in a region, then both the real part and the imaginary part separately satisfy Laplace's equation $\nabla^2\varphi = \frac{\partial^2\varphi}{\partial x^2} + \frac{\partial^2\varphi}{\partial y^2} = 0$, and furthermore the curves of constant $\operatorname{Re} f(z)$ and those of constant $\operatorname{Im} f(z)$ always cut orthogonally, so if one set is the equipotentials, the other set is field lines, meaning curves always pointing in the direction of the field.

Typically, we want to solve $\nabla^2\varphi = 0$ in a region subject to φ having specified values on the boundary. So we have to find an analytic function whose real (or imaginary) part has those values on the boundary. This is a very efficient method for some important cases (including examples solved by Jackson using more laborious techniques). The catch is, of course, finding the function: for some cases (interior of a circle) there are general methods, but in practice the cases we discuss were first solved by someone already knowing complex plane properties of familiar functions, such as the tangent.

We'll now consider some specific geometries: corners, circular pipes, rectangular ducts.

Complex Variables in Corners

This is a "real life" electrostatic problem well-represented by simple complex variable functions.

Recall first how we looked at the real and imaginary parts of the analytic function z^2, the imaginary part was $2xy$, constant on a family of hyperbolae having the axes as asymptotes. In particular, it was zero on the axes, the degenerate hyperbola. This means these hyperbolae are the equipotentials for some field in a corner between two sheets of grounded conductor at right angles. The *field lines* go out perpendicular to the surface, as usual, except at one point—the corner itself, where two equipotentials meet at right angles. The field line there comes out straight, bisecting the angle.

This would be the general appearance of the field from a point charge far out along the 45 degree line. In fact, we could do that using images, this is left as an exercise. Obviously, this field is only exact in the limit that the point charge is taken to infinity, appropriately increasing its magnitude as it moves away.

What about two grounded sheets at 45 degrees? Clearly $\operatorname{Im} z^4$ will work for the equipotentials, z^4 having zero imaginary part on both sheets. The equipotentials now are on curves of constant $\operatorname{Im} z^4 = 4xy(x+y)(x-y)$. (Note where the zero equipotentials are.) Again, this must approximate the field near the corner from a faraway charge.

But z^4 is also a solution for the ninety degree case! How could that be? Suppose we have two equal magnitude charges out along the $\pi/8$ and $3\pi/8$ lines, this would give the right shape.

How about a general angle β ? Well, $z^{\pi/\beta}$ would work—so would $z^{m\pi/\beta}$, with m an integer. In general, the potential would be a series of such terms, dominated by the first close in to the corner (because it's the lowest power) but needing the others to fit more distant boundary requirements.

We'll now look at all this from a slightly different, but equivalent, point of view: analyzing $\nabla^2\varphi = 0$ in the usual cylindrical coordinate representation.

Electrostatic Fields in Two-Dimensional Corners

We're dealing with $\nabla^2\varphi = 0$ in two dimensions, taking (ρ,ϕ) as the coordinates,

$$\frac{1}{\rho}\frac{\partial}{\partial\rho}\left(\rho\frac{\partial\varphi}{\partial\rho}\right)+\frac{1}{\rho^2}\frac{\partial^2\varphi}{\partial\phi^2}=0$$

and take separable solutions

$$\varphi(\rho,\phi)=R(\rho)\Psi(\phi)$$

to find

$$\frac{1}{R}\frac{d^2R(\rho)}{d\rho^2}=\nu^2,\quad \frac{1}{\Psi}\frac{d^2\Psi(\phi)}{d\phi^2}=-\nu^2,$$

having solutions

$$R(\rho)=a\rho^{\nu}+b\rho^{-\nu},\quad \Psi(\phi)=A\cos\nu\phi+B\sin\nu\phi.$$

How does these solutions translate to the complex number approach? They're just real and imaginary parts of simple powers, $z^{\nu}, z^{-\nu}$! (Remember $z^{\nu}=r^{\nu}e^{i\nu\theta}=r^{\nu}(\cos\nu\theta+i\sin\nu\theta)$.) There is also a special $\nu=0$ solution,

$$R(\rho)=a_0+b_0\ln\rho$$
$$\Psi(\phi)=A_0+b_0\phi$$

Assuming no charge singularity at the origin, we'll look at the imaginary part of powers of z. It's zero on the real axis for all powers but also we need it zero on the top edge of the wedge at $\phi=\beta$. That is, z^{β}

must be real, so the appropriate powers are $z^{m\pi/\beta}$, m an integer. The imaginary part of a series of such terms, plus a constant, has the form

$$\varphi(\rho,\phi)=V+\sum_{m=1}^{\infty}a_m\rho^{m\pi/\beta}\sin(m\pi\phi/\beta).$$

Near the corner, the $m=1$ term will dominate.

Exercise: try plotting the dominant first term, say for $\beta=\pi/5$, first a few equipotentials, then use those to plot field lines. (Check that your field lines approach the conductor correctly!)

The charge densities on the sides are given by the normal components of the electric fields, that is, the components in the ϕ direction,

$$E_\phi(\rho,\phi)=-\frac{1}{\rho}\frac{\partial\varphi}{\partial\phi}=-\frac{\pi a_1}{\beta}\rho^{(\pi/\beta)-1}\cos(\pi\phi/\beta).$$

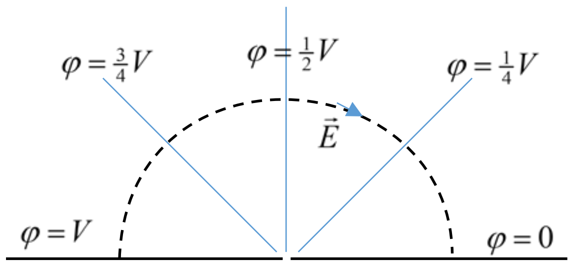
This will of course give the same sign charge density on both plates, that is, $-\frac{\varepsilon_0\pi a_1}{\beta}\rho^{(\pi/\beta)-1}$.

Notice that if $\beta=\pi$ the charge density is *independent of* ρ near the corner—not surprising, because there isn't a corner, we've flattened it!

More interesting, if $\beta>\pi$ the charge density becomes *infinite* at the corner. The extreme case is $\beta=2\pi$, just a sharp edge of a conductor, where $\sigma\sim\rho^{-1/2}$. This is still a finite amount of charge on integrating in from the edge, though. Obviously, the electric field is large near the point, depending on the width of the thin gap along the y-axis.

Exercise: Here's a $\beta=\pi$ (flattened) related problem: suppose the positive real axis has $\varphi=0$ and the negative real axis has $\varphi=V$, with no charges in the upper half plane. What is φ in the upper half plane? (*Answer on next page*.)

Answer: It's clearly $\varphi(z) = V\phi/\pi$, or $\varphi(z) = (V/\pi)\,\mathrm{Im}\ln z$.

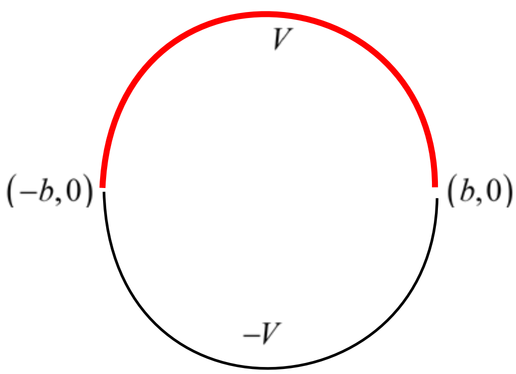


Infinite conducting sheet in (x, y) plane, thin gap along y axis. Sheet $x < 0$ at potential V, sheet $x > 0$ grounded.

There is another $\ln z$ solution: take the potential to be the real part. This is of course just a line charge at the origin, the dotted field line in the diagram becomes an equipotential in this "dual" problem.

Some Circular Equipotentials

Of course we get circular equipotentials from a line of charge, but more interesting is the case where the circle is in segments at different potentials. The simplest case has the top and bottom halves of a circular pipe at different potentials, say $\pm V$.

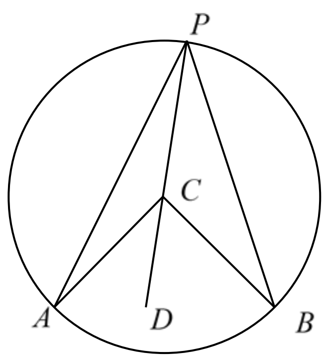


Clearly, the "equator" is an equipotential at $\varphi = 0$, and there is a very strong field where the two conductor almost meet, the equipotentials must all pass through that tiny region, and those in the top half somehow curve up from there, so at $\varphi = V$ the upper semicircle is the potential, and of course symmetric behavior in the lower half plane.

So we need to find a function analytic in the circular domain whose real (or imaginary) part has constant value V on the upper half circle, and $-V$ on the lower half circle.

The solution uses $\mathrm{Im}\ln z = i\theta$ for $z = re^{i\theta}$ plus some simple high school geometry.

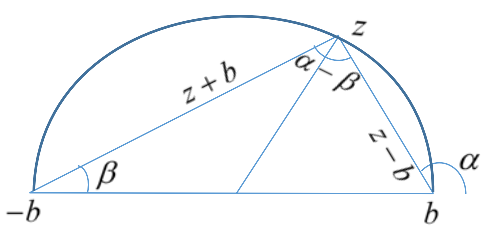
Geometry Reminder:



Given a chord AB of a circle, the angle APB for *any* point P on the circle above the chord is twice the angle ACB, where C is the center of the circle. This follows from triangles APC, BPC being isosceles, so the exterior angle ACD (which is the sum of the two other interior angles) is twice the angle APC.

We're now ready to look at Jackson problem 2.13 (incidentally copied word for word from Jeans' book, p 295 in the 1924 edition).

The key function (from hindsight, of course) is: $\varphi(z) = \operatorname{Im}\ln\left(\dfrac{z-b}{z+b}\right)$.



Write

$z - b = Ae^{i\alpha}, \quad z + b = Be^{i\beta}. \quad \ln(z-b) = \ln A + i\alpha.$

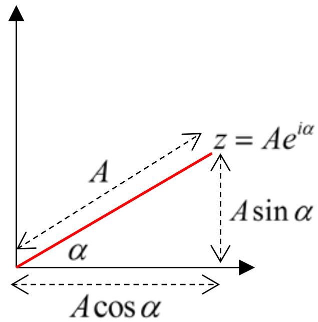
Then $\operatorname{Im}\ln\left(\dfrac{z-b}{z+b}\right) = \alpha - \beta$. Now look at that angle in the figure.

For the upper half cylinder, we know the angle in a semicircle, $\alpha - \beta = \pi/2$, so $\varphi(x, y)$ is certainly constant along this curve. Similarly, for the lower semicircle, $\alpha - \beta = -\pi/2$.

So we have the imaginary part of an analytic function, $\operatorname{Im}\ln\left(\dfrac{z-b}{z+b}\right)$, which we know from general theory satisfies $\nabla^2\varphi = 0$, we know its values on the boundary, so it's the solution to our electrostatics problem.

But there's more: our geometry theorem tells us that the other equipotentials are also *arcs of circles* through $-b, b$, they're just not semicircles.

To find an explicit expression for the potential $\varphi(x, y)$, or more naturally $\varphi(\rho, \theta)$, use that the tangent of this subtended angle is the ratio of the imaginary part to the real part,



$$Ae^{i\alpha} = A\cos\alpha + iA\sin\alpha, \quad \alpha = \tan^{-1}\left(\frac{\operatorname{Im} Ae^{i\alpha}}{\operatorname{Re} Ae^{i\alpha}}\right),$$

$$\ln Ae^{i\alpha} = \ln A + i\alpha, \quad \operatorname{Im}\ln Ae^{i\alpha} = \alpha = \tan^{-1}\left(\frac{\operatorname{Im} Ae^{i\alpha}}{\operatorname{Re} Ae^{i\alpha}}\right).$$

So, to find $\varphi(z)$ above, we need the ratio of the imaginary part to the real part of $\left(\frac{z-b}{z+b}\right)$.

Multiplying both numerator and denominator by the complex conjugate of the denominator gives a real denominator, so it plays no role in determining the *ratio* of real to imaginary parts, and the numerator becomes

$$(z-b)(z^*+b) = |z|^2 - b^2 + b(z - z^*).$$

Writing $z = \rho e^{i\theta}$ in this expression, and separating out its real and imaginary parts, we find

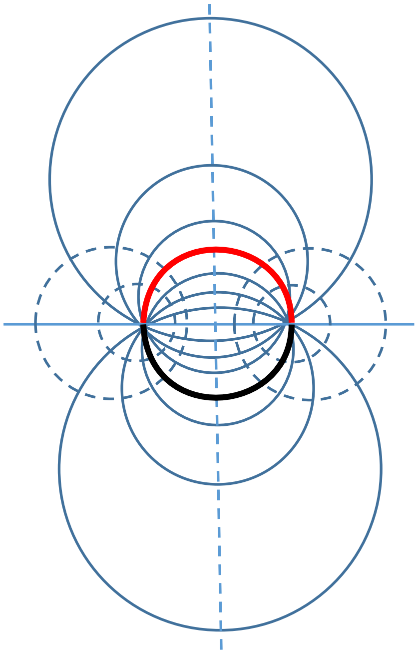
$$\operatorname{Im}\ln\left(\frac{z-b}{z+b}\right) = \tan^{-1}\left(\frac{2b\rho\sin\theta}{\rho^2 - b^2}\right).$$

(Note: this is Jackson's problem 2.13, his angle $\phi = (\pi/2) - \theta$.)

A final adjustment: Our result is equal to $\pi/2$ on the top half, $-\pi/2$ on the bottom half. But we want something equal to V_1 on the top, V_2 on the bottom. It's easy to verify the correct expression is

$$\frac{V_1 + V_2}{2} + \left(\frac{V_1 - V_2}{\pi}\right)\tan^{-1}\left(\frac{2b\rho\sin\theta}{\rho^2 - b^2}\right).$$

The equipotentials look like this (the solid circles, not the dotted ones), so it's easy to see this will work for a long horizontal cylinder with the upper half at one potential, the lower half at a different potential.

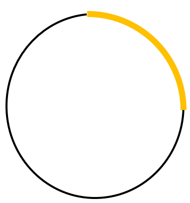


But there's more:

$$\operatorname{Re}\ln\left(\frac{z-b}{z+b}\right)=\text{constant}$$

also gives an interesting set of equipotentials: these intersect the circles we just looked at at right angles, and are in fact another set of circles, the Apollonian circles of points whose distances from $\pm b$ have a fixed ratio. These are the dotted ones: they give the equipotentials for parallel oppositely charged cylinders, as in Jackson problem 2.8. (Jeans problem 78, p 296.)

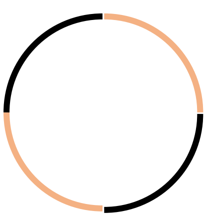
Charged Quadrant



Again we'll look at a cut circular pipe, but this time just separate off one-quarter of the circle. Suppose the quadrant (in the complex plane representation) between positive real b and pure imaginary ib is at one potential, the other three-quarters of the circle at a different potential.

The obvious strategy is to think of angles based on the chord joining the two ends b and ib, so look at $\operatorname{Im}\ln\left(\frac{z-b}{z-ib}\right)$. The angle in a quadrant is $\pi/4$ (half the angle the chord subtends at the center of the circle).

This is by way of introduction to Jackson problem 2.14(b), in which all four quadrants are electrically insulated where they join, and alternate ones are at potentials $\pm V$. We can achieve this by simply adding the opposite chord to the one above, giving the relevant function to be



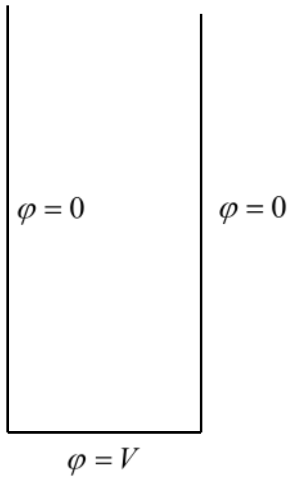
$$\operatorname{Im}\ln\left(\frac{z-b}{z-ib}\cdot\frac{z+b}{z+ib}\right)=\operatorname{Im}\ln\left(\frac{z^2-b^2}{z^2+b^2}\right).$$

As in the example above, the strategy is to multiply top and bottom by the complex conjugate of the denominator, then take arctan of the ratio of imaginary and real parts of the consequent numerator.

Exercise: try plotting the field lines just from a few facts: what do they look like near the joins? What about along the lines $x=\pm y$?

After you've sketched it, compare with this.

Rectangular Three-Sided Duct



Suppose a duct is open on one side, or rather that the two neighboring sides go to infinity in our approximation, and these sides are kept at zero potential. The bottom side in the diagram is kept at constant potential $\varphi = V$.

This is a geometry for which we can use $\tan\frac{\pi z}{2} = \frac{1}{i}\cdot\frac{e^{i\pi z}-1}{e^{i\pi z}+1}$.

Notice this function is real on the real axis, pure imaginary on the lines $x = 0, \pm 1, \pm 2, \ldots$

Therefore, if we just look at *the imaginary part of its logarithm*, $\operatorname{Im}\ln\tan\frac{\pi z}{2}$, this is zero on the real axis, and equal to $-\pi/2$ on the positive imaginary axis, and on $x = 1, y > 0$.

You might think it should have the opposite sign on that line, but to get from the real axis to these two vertical lines means going a quarter of the way around a zero at the origin, going a quarter way round the other direction a pole at $x = 1$, the phase changes are the same.

So the function we need is

$$\varphi(z) = \frac{2V}{\pi}\left(\frac{\pi}{2} + \operatorname{Im}\ln\tan\frac{\pi z}{2}\right).$$

To find the potential at a particular point (x, y) we write $\operatorname{Im}\ln\tan\frac{\pi z}{2} = \operatorname{Im}\ln\frac{1}{i}\cdot\frac{e^{i\pi z}-1}{e^{i\pi z}+1}$. The $\ln\frac{1}{i} = -\frac{\pi}{2}$ cancels the first term, leaving $\operatorname{Im}\ln\frac{e^{i\pi z}-1}{e^{i\pi z}+1}$.

The potential at (x, y) is therefore $2V/\pi$ multiplied by the phase of $\frac{e^{i\pi z}-1}{e^{i\pi z}+1}$. To find this, we multiply both top and bottom by the complex conjugate of the denominator, then the tangent of the phase is just the ratio of imaginary part/real part in the numerator, which is

$$\left(e^{i\pi(x+iy)}-1\right)\left(e^{-i\pi(x-iy)}+1\right) = e^{-2\pi y} + e^{i\pi x}e^{-\pi y} - e^{-i\pi x}e^{-\pi y} - 1,$$

from which

$$\varphi(x, y) = \frac{2V}{\pi}\tan^{-1}\left(\frac{\sin\pi x}{\sinh\pi y}\right).$$

Exercise: use this approach for a rectangular pipe, the sides at $\varphi = 0$, the top and bottom at $\varphi = V_0$.

Series Treatment Following Jackson

This is two dimensional electrostatics, so if we begin by looking for separable solutions of $\nabla^2\varphi = 0$,

$$\varphi(x,y) = X(x)Y(y)$$

then

$$\frac{1}{X(x)}\frac{d^2X(x)}{dx^2} = -\alpha^2, \quad \frac{1}{Y(y)}\frac{d^2Y(y)}{dy^2} = \alpha^2.$$

We've chosen the sign deliberately—clearly the potential is going to drop off on the y direction.

Given that $\varphi(x,y) = 0$ for $x = 0,1$ independent of y, the potential is some sum over a basic set of solutions

$$\varphi(x,y) = \sum_{n=1}^{\infty} a_n e^{-n\pi y}\sin n\pi x.$$

This already satisfies the side boundary conditions, and behaves appropriately for large y, and at $y = 0$ we must require

$$\sum_{n=1}^{\infty} a_n \sin n\pi x = V.$$

Therefore, using $\int_0^1 \sin n\pi x \sin m\pi x dx = \frac{1}{2}\delta_{mn}$, $a_n = 4V/\pi n$, but only even n's contribute, since V is symmetric about the midpoint. This gives the infinite series

$$\varphi(x,y) = \frac{4V}{\pi}\sum_{n \text{ odd}}^{\infty} \frac{1}{n} e^{-n\pi y}\sin n\pi x,$$

which Jackson proves is equivalent to our result from complex variable.

Jackson's approach has the virtue of being *much more general*: we could specify any reasonable potential $V(x)$ along the bottom plate and analyze it using this Fourier series approach.

Conformal Mapping

Conformal mapping is really just another term for what we've been doing above—but expressed rather differently. It's formulated in terms of mapping from the usual complex plane $z = x + iy$ to another complex plane $w = u + iv$, by means of a function analytic (differentiable) in the region of interest,

$$w = f(z).$$

For z_0 in the region, and $w_0 = f(z_0)$, for small deviations within the region

$$w - w_0 \cong \left(\frac{df}{dz}\right)(z - z_0) = Ae^{i\alpha}(z - z_0).$$

So locally the grid of orthogonal x, y lines goes to a grid of orthogonal u, v lines, scaled by a factor A and rotated through an angle α, but the important point is it's still an *orthogonal* grid, the right angles are preserved—this is what is meant by a *conformal* mapping. Notice in the above equation, if the small number $z - z_0$ is turned through 90 degrees, then $w - w_0$ will correspondingly turn through 90 degrees.

Conformal mappings and invariance are very common in the theory of phase transitions, string theory, etc. String theory is actually a two-dimensional theory.

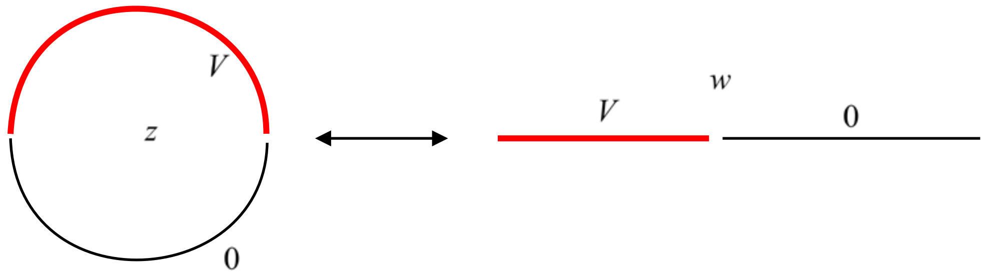
This means that if we can solve a 2D electrostatic problem for some configuration of conductors in the z plane, making an analytic transformation to the w plane automatically solves the potential problem for the transformed shapes of conductors.

A simple example is a mapping from the real axis to the unit circle, taking as usual $z = x + iy$ and $w = u + iv$,

$$z = \frac{w - i}{w + i}.$$

This maps the real w axis on to the unit circle $|z| = 1$. Check that $w = \pm\infty$ goes to $z = 1$, $w = 0$ goes to $z = -1$, and $w = \pm 1$ to $z = \mp i$. Note also that $w = -i\dfrac{z+1}{z-1}$.

Let's see how this relates the 2D problem of infinite conducting plates in the w plane, with potential zero for $u > 0$ and V for $u < 0$ to the split cylinder in the z plane with potentials V and zero for top and bottom halves respectively.



The correct potential for the w plane problem (as already discussed) is $\varphi_w(w) = (V/\pi)\operatorname{Im}\ln w$.

To try to reduce confusion, we will (temporarily) put a subscript on the potential to remind us which picture we're looking at.

In the z plane we define the potential $\varphi_z(z)$ by $\varphi_z(z) = \varphi_w(w(z))$.

That is, under the mapping $w \to z$, $\varphi_z(z)$ is assigned the same value $\varphi_w(w)$ had at the corresponding point w, meaning

$$\varphi_z(z) = (V/\pi)\operatorname{Im}\ln w = (V/\pi)\operatorname{Im}\ln\left(-i\frac{z+1}{z-1}\right).$$

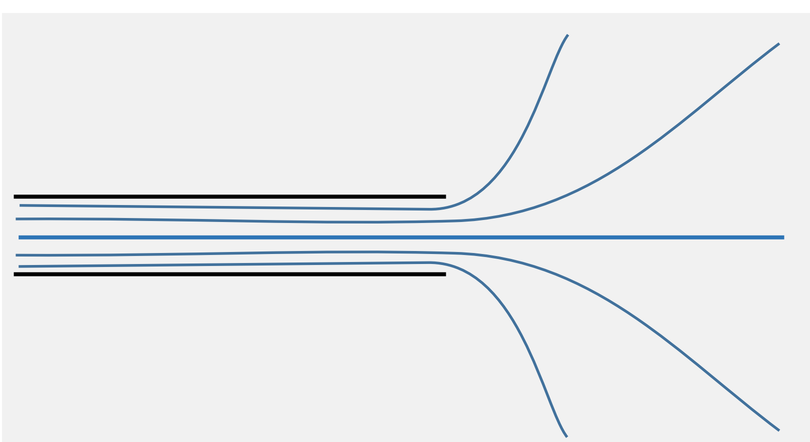
Equipotentials of course must map into equipotentials: check one or two for yourself. For example, what about the real z axis?

Maxwell's Mapping for a Parallel Plate Capacitor Edge

What makes the mapping approach worthwhile is well illustrated by Maxwell's use of it to solve an important problem: what, exactly, does the field look like at the *edge* of a parallel plate capacitor?

Maybe like this?

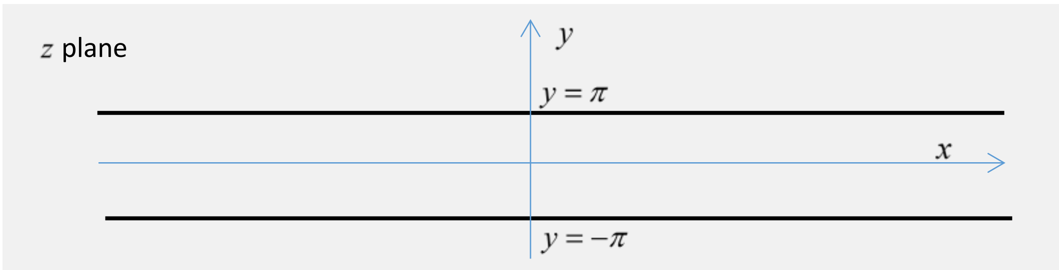
There are two distinct regions:



1. Deep inside (far to the left), the field must look like that between infinite plates—far enough away, the fact that the plates come to an end can make little difference.
2. On the other hand, well *outside* the plates, meaning at distances much greater than the distance between the plates, it will look like the field from plates at potentials $0, V$ coming together at an angle very close to 2π. We discussed this two-plates-at-an-angle configuration earlier:

for angle 2π, the equipotentials are lines radiating out from the join, the (orthogonal) electric field lines tend to concentric circles far away.

Maxwell began with infinite parallel plates



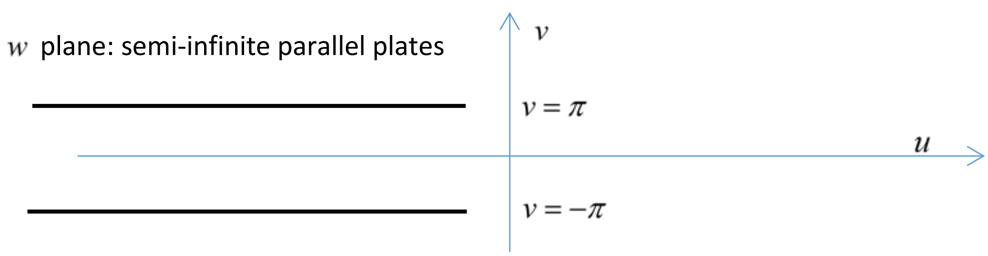
Taking the plate separation conveniently equal to 2π, he used the transformation

$$w = z + e^z.$$

In terms of real and imaginary parts,

$$u = \left(x + e^x \cos y\right), \quad v = \left(y + e^x \sin y\right).$$

Notice that this has the features we want: for x large and negative, it tends to $w = z$, meaning that moving to the left we find a field tending to the field for infinite plates, whereas for large positive x, the $z \to w$ map gives equipotentials radiating out from the end of the plates. It may be helpful to visualize the inviscid fluid analogy, coming out from the end of a pipe (of course, in 2D).



The central line, $y = 0$, goes to $v = 0$, we have a symmetric mapping, but it's not linear—look how u relates to x, pretty linear for u negative, but really taking off for u positive and increasing!

Now let's look at nonzero positive v. Not much going on for v negative, the exponential terms are tiny, but when x becomes positive and increasing ($u > -1$), the exponential term finally takes over, and as x continues to increase, the equipotential tends to a straight line at an angle v to the axis. As we approach the upper plate, this angle tends to π, so the lines are fanning out in all directions, so we've mapped the region between the plates in the z plane to the whole w plane! In particular, the equipotential infinitesimally below the top plate turns through π at the end of the plate, and goes back to $-\infty$ infinitesimally *above* the plate. *Exercise*: draw these equipotentials! *Trivia note*: Maxwell actually drew these equipotentials himself, and so well that his drawing is used as the cover for his classic book on E&M in the Dover edition.