E&M II

Part 2

Abstract

Second half of second semester of Jackson/Zangwill E&M, special relativity and radiation.

Michael Fowler

mf1i@virginia.edu

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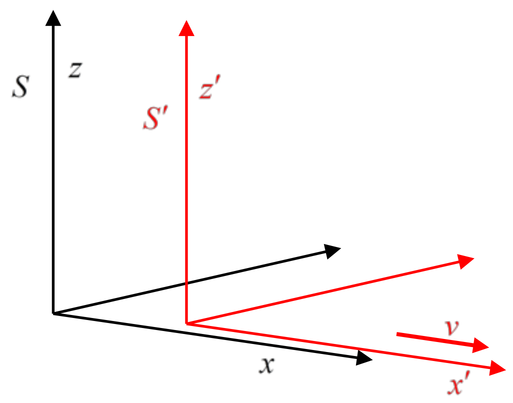
15. Special Relativity

Introduction

In the Modern Physics lectures just reviewed, we found the Lorentz transformations between two parallel frames S, S' with S' moving at constant speed v along the common x axis relative to S, both taking the zero of time to be when the origins coincide.



The result was that the coordinates (x, y, z, t) of an event in S in terms of the coordinates (x', y', z', t') of the same event in frame S' are:

$$x = \frac{x' + vt'}{\sqrt{1 - v^2/c^2}},$$
$$y = y',$$
$$z = z',$$
$$t = \frac{t' + vx'/c^2}{\sqrt{1 - v^2/c^2}}.$$

It's standard notation to write $1/\sqrt{1-(v/c)^2} = \gamma$, very common to write $(\vec{v}/c) = \vec{\beta}$, and often to take $c = 1$.

Standard Relativistic Notation

An "event" has four coordinates: position in three-dimensional space, plus time. That is, it's just a point in *four*-dimensional space, a.k.a. space time. The standard notation is

$$(ct, x, y, z) \equiv (x^0, x^1, x^2, x^3).$$

Matrix Form of Lorentz Transformation

The Lorentz transformation equations from frame S', moving at v in the x-direction relative to S, can be written in matrix form:

$$\begin{pmatrix} x^0 \\ x^1 \\ x^2 \\ x^3 \end{pmatrix} = \begin{pmatrix} \gamma & v\gamma & 0 & 0 \\ v\gamma & \gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} x^{0'} \\ x^{1'} \\ x^{2'} \\ x^{3'} \end{pmatrix},$$

The standard notation for this equation is: $x^\alpha = \Lambda^\alpha{}_{\beta'} x^{\beta'}$.

(*Hint*: to check the sign, take the nonrelativistic limit, $x^1 = v\gamma x^{0'} + \gamma x^{1'} \cong vt + x'$.)

Points to note: we have some indices up, some down. (This is really important in *general* relativity, less so here. There are various notations: contravariant and covariant, vectors and dual vectors, vectors and forms, etc. We'll just call them up and down indices.)

The $\Lambda^{\alpha'}{}_\beta$ are just the elements of the matrix written above. “Lambda” is used as *L* for Lorentz.

Four-Vectors: the Metric Tensor, Magnitude of a Vector

Definition of a four-vector: a set of four numbers in any inertial frame, $\vec{A} \xrightarrow[S]{} \{A^\alpha\}$, that transform from one frame to another like the coordinates of an event $\{x^\alpha\}$: that is, $A^\alpha = \Lambda^\alpha{}_{\beta'} A^{\beta'}$. Also called a *contravariant* vector.

Exercise: check that under a Lorentz transformation, $\vec{x}^2 - c^2t^2$ is invariant. In fact, this quantity is called the *magnitude* of the four-vector. Unlike most magnitudes, this one can be *negative*, or zero for a nonzero vector. To write it in terms of the new notation, we have to "square" the vector x^μ, but also include the information that the time and space contributions have opposite signs.

The way this is done is to introduce a *metric tensor*,

$$g_{\mu\nu} = \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$

(Some authors, including Jackson, have an overall minus sign! Watch out if you solve a problem using multiple sources…)
With this, the position vector (x^0, x^1, x^2, x^3) can be converted to one with *down* indices by:

$$x_\mu = g_{\mu\nu} x^\nu,$$

and we see this gives $(x_0, x_1, x_2, x_3) = (-x^0, x^1, x^2, x^3)$.

(The index can be lowered with $g^{\mu\nu}$, which is the inverse of $g_{\mu\nu}$, except that in our special relativity case, they're the same.)

Finally, the *magnitude* of the vector is written

$$x^\mu x_\mu = g_{\mu\nu} x^\mu x^\nu,$$

giving the result $\vec{x}^2 - c^2t^2$.

The Interval

One more piece of jargon: the *interval*. Since the Lorentz transformation is *linear*, and true for arbitrary space time points, the four-vector difference Δx^μ between two space time points clearly also transforms as a four vector, its magnitude is

$$\begin{aligned} ds^2 &= \Delta x^\mu \Delta x_\mu \\ &= -\left(\Delta x^0\right)^2 + \left(\Delta x^1\right)^2 + \left(\Delta x^2\right)^2 + \left(\Delta x^3\right)^2 \\ &= -c^2\left(\Delta t\right)^2 + \left(\Delta x\right)^2 + \left(\Delta y\right)^2 + \left(\Delta z\right)^2, \end{aligned}$$

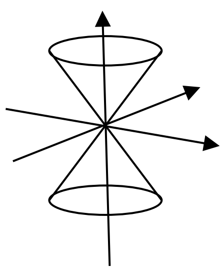
and this "square", the so-called magnitude, rather than the vector itself, is called the *interval*. Obviously, it can be positive, negative, or zero.

Spacelike, Timelike, Lightlike

Two events $\left(ct_1, x_1, y_1, z_1\right)$, $\left(ct_2, x_2, y_2, z_2\right)$ are said to be spacelike separated if the interval between them $-c^2\left(t_2 - t_1\right)^2 + \left(x_2 - x_1\right)^2 + \left(y_2 - y_1\right)^2 + \left(z_2 - z_1\right)^2 = \Delta s^2 > 0$. It is important to note that spacelike separation in one inertial frame of reference means spacelike separation in *all* inertial frames, since the magnitude is invariant under Lorentz transformation.

Similarly, timelike separation is $\Delta s^2 < 0$, lightlike separation $\Delta s^2 = 0$.

Points lightlike separated from the origin are said to be on the *light cone*, which is really two cones having vertices at the origin, the forward (in time) light cone, and the backward light cone. A light signal sent from the origin (meaning $ct = x = 0$) could trigger an event (a bomb?) anywhere on the forward light cone, a light signal from anywhere on the backward light cone could trigger an event at the origin.



An event at the origin *cannot* be the cause of another event which is outside the forward light cone.

Worldlines

As a particle moves through space time, the path traced is termed the *worldline*. Since particles travel at less than the speed of light, the world line lies within the forward light cone. A particle at rest has a worldline along the axis of the cone: in other words, the time axis. A photon has a world line on the surface of the light cone.

Relativistic Addition of Velocities

As stated above, $\left(c\Delta t, \Delta x, \Delta y, \Delta z\right)$ transforms just as $\left(ct, x, y, z\right)$ does, we'll write the transformation

$$\begin{aligned} \Delta t &= \gamma\left(\Delta t' + v\Delta x'/c^2\right) \\ \Delta x &= \gamma\left(v\Delta t' + \Delta x'\right) \\ \Delta y &= \Delta y', \quad \Delta z = \Delta z'. \end{aligned}$$

From these equations in the limit of small displacements, $\Delta x / \Delta t$ gives the *addition of velocities* formulas

$$\frac{\Delta x}{\Delta t} = u_x = \frac{\Delta x' + v\Delta t'}{\Delta t' + v\Delta x' / c^2} = \frac{u'_x + v}{1 + u'_x v / c^2} \quad \text{and} \quad u_y = \frac{u'_y}{\gamma\left(1 + u'_x v / c^2\right)}.$$

(Recall the primed frame is moving at v in the positive x-direction relative to the unprimed frame.)
A standard exercise is to consider a space station moving at v in the x-direction relative to an observer sending a rocket ship forward at u relative to the ship. What is the velocity of the rocket ship relative to the "stationary" observer? The answer is:

$$"u + v" = \frac{u + v}{1 + uv / c^2}$$

Rotations and Boosts, Rapidity

Notice now that the 4 x 4 Lorentz matrix can also represent ordinary rotations in the three-dimensional space:

$$\begin{pmatrix} x^0 \\ x^1 \\ x^2 \\ x^3 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & \cos\theta & \sin\theta & 0 \\ 0 & -\sin\theta & \cos\theta & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} x^{0'} \\ x^{1'} \\ x^{2'} \\ x^{3'} \end{pmatrix},$$

and manifestly $x^\mu x_\mu$ is invariant. Any three-dimensional rotation can be represented by the lower-right 3 X 3 minor.

In fact, the Lorentz transformation to a moving frame—called a "*boost*"—can be formulated in a strikingly similar way, in terms of a variable much favored by high energy physicists, the *rapidity* ψ, defined by

$$\frac{v}{c} = \beta = \tanh\psi, \quad \gamma = 1/\sqrt{1 - \beta^2} = \cosh\psi.$$

Rapidity proves to be a very useful parameter, because for one thing

$$\tanh\left(\psi + \psi'\right) = \frac{\tanh\psi + \tanh\psi'}{1 + \tanh\psi \tanh\psi'}$$

which is *exactly* the Lorentz addition formula for velocities! (Recall $"u + v" = \dfrac{u + v}{1 + uv / c^2}$.) This means that in successive boosts *you just add the rapidities*.

The Lorentz transformation for boosting from rest to a rapidity ψ along the x-axis is:

$$\begin{pmatrix} x^0 \\ x^1 \\ x^2 \\ x^3 \end{pmatrix} = \begin{pmatrix} \cosh\psi & \sinh\psi & 0 & 0 \\ \sinh\psi & \cosh\psi & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} x^{0'} \\ x^{1'} \\ x^{2'} \\ x^{3'} \end{pmatrix}.$$

That is, a particle at rest in the moving (boosted) frame is moving with rapidity ψ in the original frame.

Notice the similarity to the three-dimensional rotation! The sign difference ensures the transformations are unitary. Some authors (for instance, Zangwill) take time to be an *imaginary* variable, so the rotation and boost transformations look identical, but we'll stick with the more common practice. (There are in fact deep mathematical differences between rotations and boosts, as we'll see.)

*Lorentz Transformation for Arbitrary Direction

For a boost of v in the x-direction the coordinates *in the boosted frame* are:

$$\begin{pmatrix} x^{0'} \\ x^{1'} \\ x^{2'} \\ x^{3'} \end{pmatrix} = \begin{pmatrix} \gamma & -v\gamma & 0 & 0 \\ -v\gamma & \gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} x^0 \\ x^1 \\ x^2 \\ x^3 \end{pmatrix}$$

For a boost of v in the direction $\begin{pmatrix} \hat{v}_x & \hat{v}_y & \hat{v}_z \end{pmatrix}$ (the hats meaning components of a unit vector $\hat{\mathbf{v}}$) the corresponding matrix is:

$$\begin{bmatrix} t' \\ x' \\ y' \\ z' \end{bmatrix} = \begin{bmatrix} \gamma & -v_x\gamma & -v_y\gamma & -v_z\gamma \\ -v_x\gamma & 1+(\gamma-1)\dfrac{v_x^2}{v^2} & (\gamma-1)\dfrac{v_xv_y}{v^2} & (\gamma-1)\dfrac{v_xv_z}{v^2} \\ -v_y\gamma & (\gamma-1)\dfrac{v_yv_x}{v^2} & 1+(\gamma-1)\dfrac{v_y^2}{v^2} & (\gamma-1)\dfrac{v_yv_z}{v^2} \\ -v_z\gamma & (\gamma-1)\dfrac{v_zv_x}{v^2} & (\gamma-1)\dfrac{v_zv_y}{v^2} & 1+(\gamma-1)\dfrac{v_z^2}{v^2} \end{bmatrix} \begin{bmatrix} t \\ x \\ y \\ z \end{bmatrix},$$

Notice first that this does give the right answer for the boost along the x-axis.
Our strategy for boosting in an arbitrary direction is to reorient the system so that that direction becomes the x-axis, apply our known boost, then rotate it back.
To see how this works, we write the above matrix in terms of blocks, as follows:

$$\begin{pmatrix} \gamma & -\gamma\mathbf{v}^T \\ -\gamma\mathbf{v} & I+(\gamma-1)\hat{\mathbf{v}}\hat{\mathbf{v}}^T \end{pmatrix}.$$

In this same block notation, a three-dimensional rotation has the form

$$\begin{pmatrix} 1 & 0 \\ 0 & \mathbf{R} \end{pmatrix}$$

and its inverse is $\begin{pmatrix} 1 & 0 \\ 0 & \mathbf{R}^T \end{pmatrix}$. If we choose $\mathbf{R}$ such that $\mathbf{Rv}$ points along the x-axis, then

$$\begin{pmatrix} 1 & 0 \\ 0 & \mathbf{R} \end{pmatrix}\begin{pmatrix} \gamma & -\gamma\mathbf{v}^T \\ -\gamma\mathbf{v} & I+(\gamma-1)\hat{\mathbf{v}}\hat{\mathbf{v}}^T \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & \mathbf{R}^T \end{pmatrix}=\begin{pmatrix} \gamma & -\gamma\mathbf{v}^T\mathbf{R}^T \\ -\gamma\mathbf{Rv} & I+(\gamma-1)\mathbf{R}\hat{\mathbf{v}}\hat{\mathbf{v}}^T\mathbf{R}^T \end{pmatrix}$$

where $\mathbf{Rv}=\begin{pmatrix} v \\ 0 \\ 0 \end{pmatrix}, \quad \mathbf{v}^T\mathbf{R}^T=\begin{pmatrix} v & 0 & 0 \end{pmatrix}$. The reader can easily check that this is the form we found for a boost along the *x*-axis— so reversing the process gives us the general case. (And, we don't have to find the explicit form of $\mathbf{R}$!)

A Bit of Group Theory

The Lorentz boosts along the *x*-axis formed an Abelian (commutative) group, just as the set of rotations in a plane do. The rotations in a plane are a subgroup of the group of three-dimensional rotations, which is of course non-abelian. What about the set of all Lorentz boosts? It turns out that this is *not* a group. A product of two Lorentz boosts in different directions is not just a Lorentz boost in some combined direction, it also has some rotation. The Lorentz group is the group of boosts *plus* rotations.

Proper Time and Four-Velocity

Consider a spaceship going from one planet to another, the planets might have quite different velocities, so the distance covered by the ship will be different in the two planet rest frames. One thing that won't be different is the time elapsed as measured by the crew of the spaceship. This is called the *proper time* of the spaceship, the clock is always with the ship.

An increment of proper time is denoted by $d\tau$.

If the spaceship moves Δx^μ in time $d\tau$, this incremental displacement transforms as a Lorentz four-vector. Therefore, so does

$$U^\mu=\frac{dx^\mu}{d\tau}.$$

The four-vector U^μ is called the *four-velocity*. In the nonrelativistic limit it becomes (c, v^i), the spatial part just the ordinary velocity, and $\tau \to t,\ x^0 = ct$.

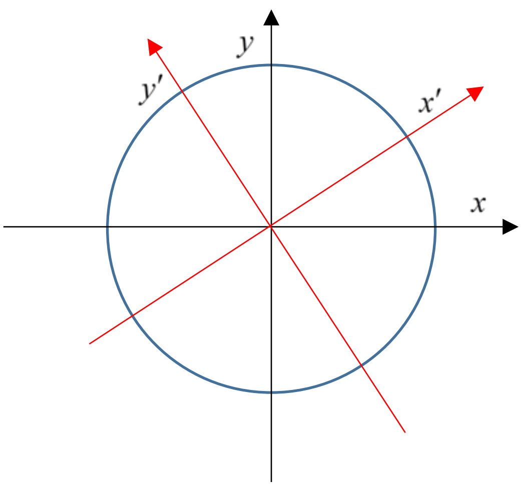
Now $U^\mu U_\mu = \dfrac{dx^\mu dx_\mu}{(d\tau)^2}$, but $dx^\mu dx_\mu$ is just the interval, which has the same value in all frames, including the frame in the ship, where it is $-(d\tau)^2$, so it follows that

$$U^\mu U_\mu = -c^2.$$

So in the rest frame, where the incremental movement along the world line dx^μ is purely in the time direction, and is just $d\tau$, the four-velocity is $(c,0,0,0)$.
In general, it's $(\gamma c, \gamma v^1, \gamma v^2, \gamma v^3)$. (In relativity papers and books, these formulas usually appear with $c = 1$.)

New Frame Axes and Scales in the Old Frame

For ordinary rotations in a plane, circles centered at the origin are invariant, and in particular the unit circle cuts the axes, old and new, at the point one, so it connects—rather trivially—the two *scales*, in the new frame and the old frame.



In order to discuss length contraction and time dilation, it is essential to have a way to connect axes and scales in different frames.

For the Lorentz transformation, instead of the simple invariant circles $x^2 + y^2 = R^2$, we evidently have invariant *hyperbolae*, $-c^2t^2 + x^2 = a^2$, or $-c^2t^2 + x^2 = -b^2$ (a, b real.)

The circles came from a rotation matrix

$$\begin{pmatrix} x' \\ y' \end{pmatrix} = \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}$$

and $\cos^2\theta + \sin^2\theta = 1$.

The invariant hyperbolae come analogously from

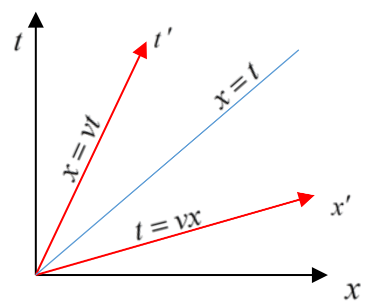
$$\begin{pmatrix} ct' \\ x' \end{pmatrix} = \begin{pmatrix} \cosh\psi & -\sinh\psi \\ -\sinh\psi & \cosh\psi \end{pmatrix} \begin{pmatrix} ct \\ x \end{pmatrix} = \mathbf{\Lambda}(\psi) \begin{pmatrix} ct \\ x \end{pmatrix}.$$

and $\cosh^2\psi - \sinh^2\psi = 1$.

Recall that $\tanh\psi = 0$ for $\psi = 0$, and $\tanh\psi \to \pm 1$ as $\psi \to \pm\infty$.

First, the lines $ct = \pm x$ must go to $ct' = \pm x'$, they constitute the two-dimensional version of the light cone.

This light cone invariance only works because there is one sign change in $\Lambda(\psi)$ compared with $R(\theta)$, and that sign change means that the t', x' axes turn in *opposite* directions from the t, x axes, in contrast to the ordinary rotation, so going to larger and larger boosts, the axes close like scissors around the line $x = t$, never reaching it, of course.



This is easy to see from the equations: the t-axis is the line $x = 0$, the t' axis is the line $x' = 0$, or $x = vt$.

Put another way: the t-axis is the “world line”, meaning the path in space time, of an object at rest at the origin in the original frame, the t'-axis is the world line of an object at rest at the origin of the primed frame.

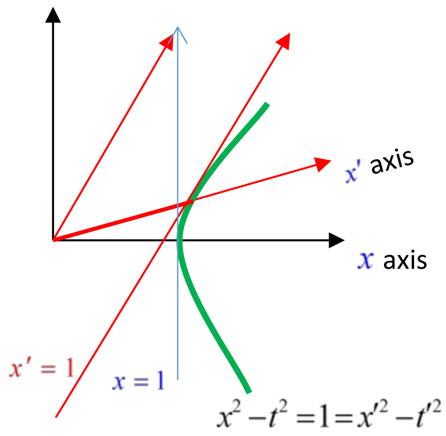
The x' axis is the line $t' = 0$, so $t = vx$ in the original frame.

So the primed frame axes are the original axes turned through *opposite* angles $\pm\theta$, $\tan\theta = v = \tanh\psi$. This means that for small speeds, θ, v, ψ are close, but as ψ goes to infinity, θ just approaches 45°.

These diagrams are often called *Minkowski diagrams*—first drawn by Minkowski a few years after Einstein published his special relativity paper.

Finding Length and Time Scales in a New Frame: Invariant Hyperbolae

(*Note*: we'll take $c = 1$ in the following sections.)



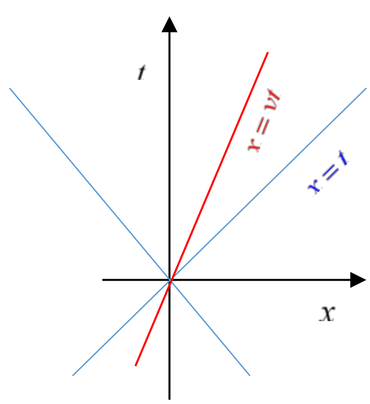
We've now seen how the axes move, but we haven't tracked what happens to the *calibration*—the scale on the axes. The way to do that is to use an invariant hyperbola, for example $-t^2 + x^2 = 1 = -t'^2 + x'^2$.
This hyperbola cuts the x-axis at $x = 1$, and the x' axis at $x' = 1$. Note that $x' = 1$ is the tangent line to the unit hyperbola $x'^2 - t'^2 = 1$, it's the minimum possible value of x' on that hyperbola.

Lorentz Contraction

Notice that from the diagram, in the (x,t) plane the point $x' = 1$ is further from the origin than the point $x = 1$. Does this mean that a rod of unit length at rest in the primed frame (say, stretching from $x' = 0$ to $x' = 1$ will appear longer than unity in the (x,t) frame?
Presumably not—that would be the *opposite* of Lorentz contraction.
So what's going on? The essential point is that we're looking at the x -positions of the ends of the rod at *different times* t. To measure the length of a moving rod, we obviously need to find the x-values of the end points *at the same time* t.

World Lines

As we mentioned earlier, the world line of a particle (or of a small part of a solid object) is its *path in four-dimensional spacetime*.

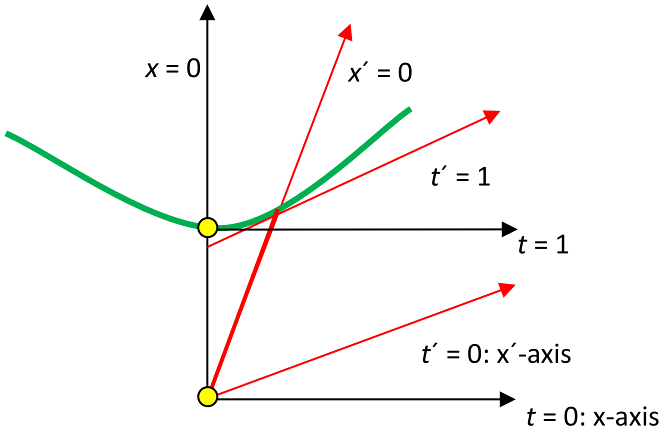


Here are some sample world lines in a two dimensional subspace. First, the light cone sections are world lines of photons, traveling at $c = 1$. A particle moving at constant velocity $x = vt$ is shown. The world line must be *steeper* than the light cone, since $v < c$. A particle *at rest* has a world line parallel to the t axis, so the t axis itself is the world line of a particle at rest at the origin.
Now, back to measuring the moving rod. We need to plot the world lines of the two ends, and find how far apart they are at, say, $t = 0$. Look back at the original diagram. The world line of the left end is just that of the primed origin, that is, it's the t' axis, $x' = 0$. The other end is moving at the same velocity, so its world line has the same slope: it's $x' = 1$, the tangent line to the unit hyperbola $x'^2 - t'^2 = 1$,as mentioned earlier.
Plotting the two world lines, we can see that their simultaneous intercepts on $t = 0$ are at *points less than one unit apart*.
Exercise: check your understanding by using similar arguments to show that a rod of unit length at rest in the original unprimed frame will have length measured as less than one in the primed frame.

Time Dilation

We've just seen how an invariant space-like hyperbola can explain how each observer can see the other as Lorentz contracted. A *time-like* invariant hyperbola can show us that each sees the other's clock as running slow.

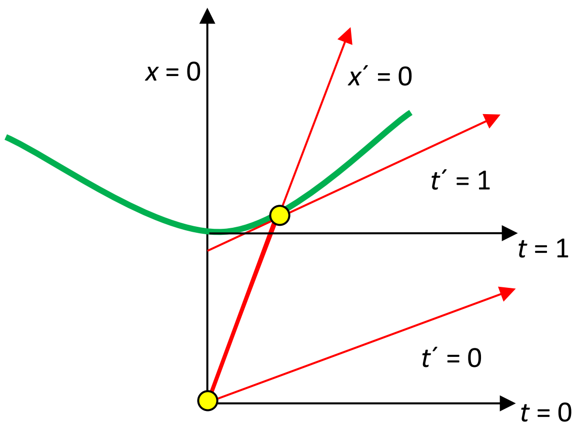
Here is the invariant hyperbola is $-t^2 + x^2 = -1 = -t'^2 + x'^2$:



Two yellow circles represent flashes one second apart at origin in *unprimed* frame

The red parallel lines here are the lines $t' = 0$ and $t' = 1$, both lines of simultaneity in the primed frame. Suppose first that a clock in the unprimed frame flashes once a second. The initial flash is seen by both observers to be at their common origin, $t = t' = 0$. The second flash, at $t = 1$, is clearly at $t' > 1$--the clock is running slow in the primed frame.

Now suppose a clock in the *primed* frame is flashing once a second. As before, the initial flash is at the common origin. The next flash, at $t' = 1, x' = 0$ is clearly at $t > 1$:



Two yellow circles represent flashes one second apart at origin in *primed* frame

Look at the invariant hyperbolae *and scale markings* on this animation!

Four-Acceleration

The *acceleration four vector* is defined as $\vec{a} = d\vec{U} / d\tau$. Notice that since the four velocity has constant magnitude $\vec{U} \cdot \vec{U} = -1$, *the four acceleration is always orthogonal to the four velocity*: $\vec{U} \cdot d\vec{U} / d\tau = 0$, and so *in the frame of the moving object the four acceleration has only spatial components*.

The acceleration four vector can also be written $a^\alpha = \dfrac{dU^\alpha}{d\tau} = \dfrac{dU^\alpha}{dx^\beta}\dfrac{dx^\beta}{d\tau} = U^\beta \partial_\beta U^\alpha$.

14. Relativistic Dynamics I

Summarized from my Modern Physics Notes

Newton's Laws Revisited

1. *Principle of Inertia*: no external forces, no change in motion.
Defining property of an inertial frame: basis of SR.

2. force = mass x acceleration, cannot be true: from the formula for addition of velocities.

3. action = reaction, implies *simultaneous* measurements, so if true in *A*'s inertial frame, it won't be in *B*'s!

Conservation Laws

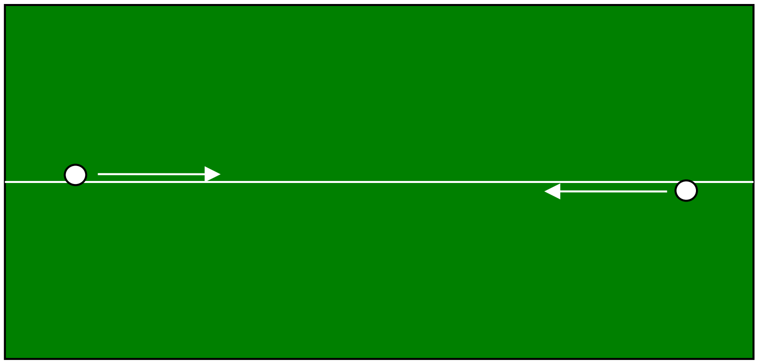
Third Law gives the *total* rate of change of momentum of the system caused by the interaction is zero. *conservation of momentum*. Independent of details of the forces

The other major dynamical conservation law is the *conservation of energy*.

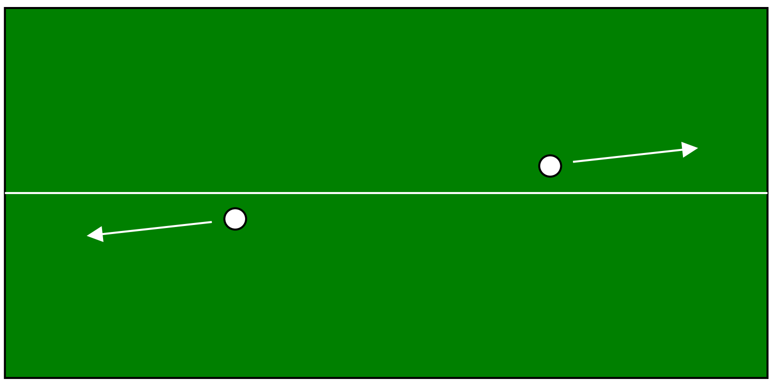
Einstein's belief: these conservation laws assumed them to be satisfied *in all inertial frames*, and explored the consequences.

Momentum Conservation on the Pool Table

Consider the following:



Balls on pool table moving towards glancing collision



Motion of balls on table after collision

We'll now move to bigger things...

A Symmetrical Spaceship Collision

In center of mass frame, y-velocities—is equal and opposite.

In A's frame, say A's y velocity is 15 meters per second. In B's frame, B's y velocity is 15 meters per second.

But what is B's y velocity as measured by A? It will be less: just as a moving clock is slowed down: $15/\gamma$ m/sec.

So how can momentum be conserved in this frame?

Einstein rescues Momentum Conservation

Assume moving masses are *heavier* by the right factor: $m = \dfrac{m_0}{\sqrt{1 - v^2 / c^2}}$

15. Relativistic Dynamics II

Continuing summarizing the Modern Physics notes...

Particle Creation: Energy into Mass

The first "new" particle created in accelerators was the pion:

$$p + p \to p + p + \pi^0,$$

and

$$p + p \to p + n + \pi^+.$$

The neutral pion mass is 135 MeV, the charged pions have mass 140 MeV, where we follow standard high energy practice in calling mc^2 the “mass”.

Energy Necessary to Produce a Pion

An incoming proton with 135 MeV of kinetic energy will not be able to create a neutral pion (rest mass 135 Mev) in a collision with a stationary proton. (Must also conserve momentum.)
Go to the center of mass frame, where initially two protons are moving towards each other with equal and opposite velocities, there being no total momentum. Obviously, in this frame the least possible K.E. must be just enough to create the π^0 with *all* the final state particles $\left(p, p, \pi^0\right)$ at rest.

$$E = \frac{2m_p c^2}{\sqrt{1 - v^2 / c^2}} = 2m_p c^2 + m_\pi c^2.$$

we find the two incoming protons must both be traveling at 0.36*c*.
In the lab frame we must add a velocity of 0.36*c* + 0.36*c* = 0.64*c*. incoming proton has a relativistic mass of 1.3 times its rest mass, and thus a K.E. around 280 MeV.
To create a pion of rest energy 135 MeV, it is necessary to give the incoming proton at least 290 MeV of kinetic energy. This is called the “threshold energy”.

Antiproton Production

$$p + p \to p + p + p + \overline{p}.$$

What about

$$p + p \to p + p + \pi^+ + \overline{p}\,?$$

Doesn’t happen—so charge conservation isn’t the only constraint on which particles can be produced.
Baryon number conserved.

A Machine Built to Produce One Particle

One of the first modern accelerators, built at Berkeley in the fifties, was designed specifically to produce the antiproton.

We can use $E^2 - c^2\vec{p}^2 = E'^2 - c^2\vec{p}'^2$ to get lab frame information from the center of mass frame.

In the center of mass (CM) frame the momentum is zero, and in the lab frame the momentum is all in the incoming proton, so

$$E_{\rm cm}^2 = ((m_{\rm in} + m_0)c^2)^2 - c^2 p_{\rm in}^2$$

where here m_0 is the proton rest mass,

$$m_{\rm in} = \frac{m_0}{\sqrt{1 - v_{\rm in\,LAB}^2 / c^2}}.$$

At the antiproton production threshold, $E_{cm} = 4m_0c^2$, so

$$16m_0^2c^4 = m_{\rm in}^2c^4 + 2m_{\rm in}c^2m_0c^2 + m_0^2c^4 - c^2p_{\rm in}^2,$$

and using

$$m_{\rm in}^2c^4 - c^2p_{\rm in}^2 = m_0^2c^4,$$

we find

$$2(m_{\rm in}c^2)(m_0c^2) + 2(m_0c^2)^2 = 16(m_0c^2)^2,$$

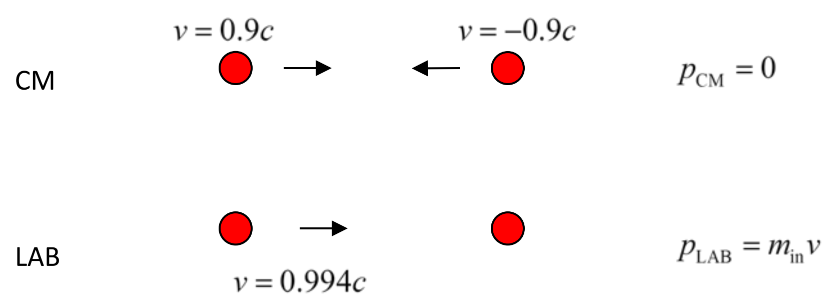
so

$$m_{\rm in}c^2 = 7m_0c^2.$$

Therefore to create two extra particles, with total rest energy $2m_0c^2$, it is necessary for the incoming proton to have a *kinetic* energy of $6m_0c^2$. The Berkeley Gevatron had design energy 6.2 GeV.

Higher Energies

As we go to higher energies, this "inefficiency" gets worse—consider energies such that the kinetic energy >> rest energy, and assume the incoming particle and the target particle have the same rest mass, m_0, with the incoming particle having relativistic mass m_{in} :



The same collision as viewed in the CM and LAB frames.

Comparing the center of mass energy with the lab energy at these high energies,

$$\begin{aligned} E_{\text{LAB}} &= \left(m_{\text{in}} + m_0\right)c^2, \\ E_{\text{CM}}^2 &= E_{\text{LAB}}^2 - p_{\text{LAB}}^2 c^2 \\ &= m_{\text{in}}^2 c^4 + 2m_{\text{in}} c^2 m_0 c^2 + m_0^2 c^4 - p_{\text{LAB}}^2 c^2 \\ &= 2m_0 c^2 (m_{\text{in}} c^2 + m_0 c^2). \end{aligned}$$

For $m \gg m_0$,

$$E_{CM}^2 \approx 2m_0 c^2 mc^2 \approx 2m_0 c^2 . E_{LAB}$$

so

- $$E_{CM} \approx \sqrt{2m_0 c^2 . E_{LAB}} \,,$$

ultimately one must *quadruple* the lab energy to **double t**he center of mass energy. And, at higher energies, things get steadily worse—this is why colliders were built!

Principle of Least Action

(*Following Landau*)

For any mechanical system there is an integral $S,$ called the action, which has a minimum value for the path traced in configuration space as the system evolves in time, as opposed to other paths between the initial and final configurations.
The action integral for a free material particle cannot depend on the coordinate system chosen, it must be Lorentz invariant. The only possibility for a free particle with no external force is integrating over the infinitesimal intervals

$$S = -\alpha \int_a^b c d\tau,$$

with τ the proper time, and α some yet to be determined constant. Since we know a moving clock runs slow, we see that in the inertial frame in which the beginning event a and the final event b are both at the spatial origin, the path corresponding to staying at that origin is the extremum. Actually it's the *longest* possible time, so we put in the minus sign.
The standard notation for the action in mechanics is an integral over time,

$$S = \int_{t_1}^{t_2} L dt,$$

where L is the Lagrangian. Since we know that $d\tau = \gamma dt,$ evidently

$$L = -\alpha c\sqrt{1-\frac{v^2}{c^2}} \approx -\alpha c + \frac{\alpha v^2}{2c}.$$

We know that non-relativistically, $L = \frac{1}{2}mv^2,$ and the first constant $-\alpha c$ is irrelevant in minimizing, so we have

$$S = -mc^2 \int_a^b d\tau,$$

– and

$$L = -mc^2\sqrt{1-\frac{v^2}{c^2}}.$$

The momentum

$$\vec{p} = \frac{\partial L}{\partial \vec{v}} = \frac{m\vec{v}}{\sqrt{1-v^2/c^2}}.$$

The energy is defined as

$$E = \vec{p} \cdot \vec{v} - L = \frac{mc^2}{\sqrt{1 - v^2 / c^2}}.$$

The principle of least action is that along the physical path

$$\delta S = -mc \int_a^b d\tau = 0.$$

Writing $d\tau = \sqrt{-dx_i dx^i}$ (always real, a particle must follow a timelike path) we have

$$\delta S = mc \int_a^b \frac{dx_i \delta dx^i}{d\tau} = mc \int_a^b u_i d\delta x^i,$$

and integrating by parts

$$\delta S = -mcu_i \delta x^i \Big|_a^b - mc \int_a^b \delta x^i \frac{du^i}{d\tau} d\tau.$$

Taking arbitrary δx^i, except zero at the ends, we find the four-acceleration $du^i / d\tau = 0$ along the path, as we expect.
We'll return to this result after discussing how electromagnetic fields Lorentz transform, then we'll include the interaction with a charged particle, to find the path in a field.

Mass Really *Does* Increase with Speed:

Accelerator design is based on this.

...Or Does It?

Actually, there is continuing debate among physicists concerning this concept of relativistic mass. The debate is largely semantic: no-one doubts that the correct expression for the momentum of a particle having a rest mass m moving with velocity $\vec{v}$ is $\vec{p}=\frac{m}{\sqrt{1-v^2/c^2}}\vec{v}$.

Mass and Energy Conservation: Kinetic Energy and Mass for Very Fast Particles

Newton's Second Law for variable mass is

$$F = dp/dt,$$

Close to the speed of light,

$$F=\frac{dp}{dt}=\frac{d(mv)}{dt}\cong\frac{dm}{dt}c$$

In time dt it will move cdt and the force will do work $Fcdt$.

$$Fcdt=(dm)c^2.$$

So the energy dE expended by the accelerating force in the time dt yields an increase in mass, and $dE=(dm)c^2$.

Kinetic Energy and Mass for Slow Particles

$$\sqrt{1-v^2/c^2}\cong 1-\tfrac{1}{2}v^2/c^2$$

and

$$\frac{1}{1-\frac{1}{2}v^2/c^2}\cong 1+\tfrac{1}{2}v^2/c^2.$$

So, for $v/c \ll 1$,

$$m(v) \cong m_0\left(1+\tfrac{1}{2}v^2/c^2\right)$$
$$dm \cong \left(\tfrac{1}{2}m_0v^2\right)/c^2 = KE/c^2.$$

Again, the mass increase dm is related to the kinetic energy KE by $KE = (dm)c^2$. Having looked at two simple cases, we're ready to derive the general result, valid over the whole range of possible speeds.

Kinetic Energy and Mass for Particles of Arbitrary Speed

Left as an exercise.

Einstein's Box

An amusing "experiment" on the equivalence of mass and energy is the following: consider a closed box with a flashlight at one end and light-absorbing material at the other end. Imagine the box to be far out in space away from gravitational fields or any disturbances. Suppose the light flashes once, the flash travels down the box and is absorbed at the other end.

Now it is known from Maxwell's theory of electromagnetic waves that a flash of light carrying energy E also carries momentum $p = E/c$. Thus, as the flash leaves the bulb and goes down the tube, the box recoils, like a gun, to conserve overall momentum. Suppose the whole apparatus has mass M and recoils at velocity v. Of course, $v \ll c$.

Then from conservation of momentum in the frame in which the box was initially at rest:

$$Mv = E/c,$$

the recoil momentum of the box equals (minus) the momentum of the flash emitted.

After a time $t = L/c$ the light hits the far end of the tube, is absorbed, and the whole thing comes to rest again. (We are assuming that the distance moved by the box is tiny compared to its length.)

How far did the box move? It moved at speed v for time t, so it moved distance

$$d = vt = vL/c.$$

From the conservation of momentum equation above, we see that $v = E/Mc$, so the distance d the box moved over is:

$$d = \frac{EL}{Mc^2}.$$

Now, the important thing is that there are *no* external forces acting on this system, so *the center of mass cannot have moved*!

The only way this makes sense is to say that to counterbalance the mass M moving d backwards, the light energy *must have transferred a small mass* m, say, the length L of the tube so that

$$Md = mL$$

and balance is maintained. From our formula for d above, we can figure out the necessary value of m,

$$m = \frac{M}{L}d = \frac{M}{L}\frac{EL}{Mc^2} = \frac{E}{c^2}$$

so

$$E = mc^2.$$

We have therefore established that *transfer of energy implies transfer of the equivalent mass*. Our only assumptions here are that the center of mass of an isolated system, initially at rest, remains at rest if no external forces act, and that electromagnetic radiation carries momentum E/c, as predicted by Maxwell's equations and experimentally established.

How Does the Total Energy of a Particle Depend on Momentum?

$$E = mc^2 = \frac{m_0c^2}{\sqrt{1 - v^2/c^2}},$$

The momentum varies with speed as

$$p = mv = \frac{m_0v}{\sqrt{1 - v^2/c^2}}.$$

Now

$$E^2 = m^2c^4 = \frac{m_0^2c^4}{1 - v^2/c^2}$$

so

$$m^2c^4(1-v^2/c^2)=m_0^2c^4$$
$$m^2c^4-m^2v^2c^2=m_0^2c^4$$
$$m^2c^4=E^2=m_0^2c^4+m^2c^2v^2$$

hence using $p=mv$ we find

$$E=\sqrt{m_0^2c^4+c^2p^2}\ .$$

If p is very small, this gives

$$E\approx m_0c^2+\frac{p^2}{2m_0},$$

the usual classical formula.

If p is very large, so $c^2p^2\gg m_0^2c^4$, the approximate formula is $E=cp$.
The High Kinetic Energy Limit: Rest Mass Becomes Unimportant!

Transforming Energy and Momentum to a New Frame

We have shown

$$\vec{p}=m\vec{v}=\frac{m_0\vec{v}}{\sqrt{1-v^2/c^2}}$$

$$E=mc^2=\sqrt{m_0^2c^4+c^2\vec{p}^2}.$$

Notice we can write this last equation in the form

$$E^2-c^2\vec{p}^2=m_0^2c^4..$$

That is to say, $E^2-c^2\vec{p}^2$ depends *only* on the rest mass of the particle and the speed of light. It does not depend on the velocity of the particle, so it must be the same—for a particular particle—in all inertial frames.

This is reminiscent of the invariance of $\vec{x}^2-c^2t^2$, the interval between two events, under the Lorentz transformations. One might guess from this that the laws governing the transformation from E,p in one Lorentz frame to E',p' in another are similar to those for t,x. We can actually derive the laws for E,p to check this out.
As usual, we consider all velocities to be parallel to the x-axis.

We take the frame S' to be moving in the x-direction at speed v relative to S.

Consider a particle of mass m_0 (rest mass) moving at u' in the x' direction in frame S', and hence at u along x in S, where

$$u = \frac{u' + v}{1 + vu'/c^2}.$$

The energy and momentum in S' are

$$E' = \frac{m_0 c^2}{\sqrt{1 - u'^2/c^2}}, \quad p' = \frac{m_0 u'}{\sqrt{1 - u'^2/c^2}}$$

and in S:

$$E = \frac{m_0 c^2}{\sqrt{1 - u^2/c^2}}, \quad p = \frac{m_0 u}{\sqrt{1 - u^2/c^2}}.$$

It is straightforward (and you should do it!) to show that

$$E = \frac{1}{\sqrt{1 - v^2/c^2}}\left(E' + vp'\right).$$

Similarly, we can show that

$$p = \frac{p' + vE'/c^2}{\sqrt{1 - v^2/c^2}}.$$

Lorentz transformations for particle energy and momentum. It follows that

$$E^2 - c^2 p^2 = E'^2 - c^2 p'^2 = m_0^2 c^4.$$

Energy, Momentum and Four-Velocity

Recall now from the previous lecture the definition of four velocity in terms of an infinitesimal interval and the proper time elapsed:

$$U^{\mu} = \frac{dx^{\mu}}{d\tau}.$$

In the nonrelativistic limit it becomes $\left(c, v^{i}\right)$, the spatial part just the ordinary velocity, and $\tau \to t,\ x^{0} = ct.$

Now $U^{\mu}U_{\mu} = \dfrac{dx^{\mu}dx_{\mu}}{\left(d\tau\right)^{2}}$, but $dx^{\mu}dx_{\mu}$ is just the interval, which has the same value in all frames, including the frame in the ship, where it is $-\left(d\tau\right)^{2}$, so it follows that

$$U^{\mu}U_{\mu} = -c^{2}.$$

So in the rest frame, where the incremental movement along the world line dx^{μ} is purely in the time direction, and is just $d\tau,$ the four-velocity is $\left(c,0,0,0\right).$
In general, it's $\left(\gamma c, \gamma v^{1}, \gamma v^{2}, \gamma v^{3}\right).$
Hence the energy (including rest energy) and momentum together can be written

$$\left(E/c, \vec{p}\right) = m_{0}U^{\mu}.$$

So $\left(E/c, \vec{p}\right)$ must transform just as dx^{μ} does.

Photon Energies in Different Frames

For a zero rest mass particle, such as a photon, $E = cp, \quad E^{2} - c^{2}p^{2} = 0$ in all frames.

Thus

$$E = \frac{E' + vp'}{\sqrt{1 - v^{2}/c^{2}}} = \frac{E' + vE'/c}{\sqrt{1 - v^{2}/c^{2}}} = E'\sqrt{\frac{1 + v/c}{1 - v/c}}.$$

Since $E = cp,\ E' = cp'$ we also have

$$p = p'\sqrt{\frac{1 + v/c}{1 - v/c}}.$$

Notice that the ratios of photon *energies* in the two frames coincides with the ratio of photon *frequencies* found in the Doppler shift.

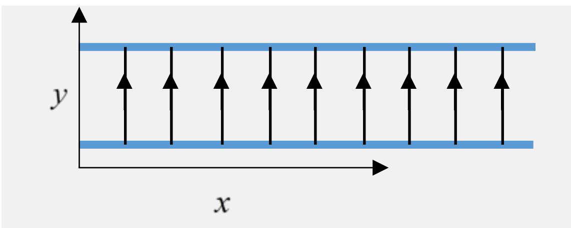
16. Special Relativity: Electromagnetism

(*Like Griffiths, I follow Purcell.*) *4/3/18*

How Fields Transform

The whole point of using electric and magnetic fields is that if the electric field is defined in terms of the electrical force on a stationary charge, then it *only* depends on that local value $\vec{E}(\vec{r})$, not on whether the field originated from another close-by point charge, or a distant larger one, or even a changing magnetic field. An electric field is an electric field.

This means that to see how electric fields transform under Lorentz transformations, we can take the simplest possible way of producing the field, something easy to analyze in different frames. The obvious candidate is a parallel plate capacitor, with infinite plates, a distance d apart in their rest frame, having uniform surface charge density σ.



The electric field between the plates is $\vec{E} = \frac{\sigma}{\varepsilon_0}\hat{\vec{y}}$.

Suppose now the plates are in a frame S', and frame S' is moving in the x-direction at speed v relative to our lab frame S. That is, the direction of motion is *perpendicular* to the direction of the field.

Then from the Lorentz contraction of the plates, and the invariance of electric charge under a Lorentz transformation (we're assuming this, but it's extremely well-verified experimentally), the surface charge density increases by a factor γ :

$$\sigma \to \sigma' = \frac{\sigma}{\sqrt{1-v^2/c^2}} = \gamma\sigma,$$

And therefore so does the electric field strength:

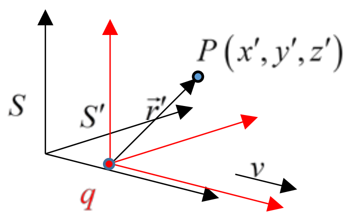
$$\vec{E}'_\perp = \gamma \vec{E}_\perp .$$

The subscript $\perp$ is to make clear that this is the change in field strength in a transformation to a frame moving in a direction *perpendicular* to the field direction.

How does motion in a direction *parallel* to the field affect it? Not at all: the distance between the plates will shrink, but the plate surface charge density in unaffected, so the field doesn't change:

$$\vec{E}'_{\parallel} = \vec{E}_{\parallel} .$$

Electric Field of a Point Charge in Uniform Motion



The equations above for $\vec{E}_\perp, \vec{E}_\parallel$ give us enough information to construct the electric field of a moving point charge. Let's assume the charge q is at rest at the origin of the frame S', but that S' is moving to the right along the x-axis at v relative to the parallel lab frame S. What does the electric field look like in S? We know that in S', at the point P with coordinates (x', y', z') the field is

$$\vec{E}' = \frac{1}{4\pi\varepsilon_0}\frac{q}{r'^3}\vec{r}' = \frac{1}{4\pi\varepsilon_0}\frac{q}{\left(x'^2 + y'^2 + z'^2\right)^{3/2}}(x', y', z'),$$

so in a frame S moving relative to S' at velocity $-v$ along the x axis, the y, z components will be enhanced by a factor γ, the x component won't change, so in frame S:

$$\vec{E}_S = \frac{1}{4\pi\varepsilon_0}\frac{q}{\left(x'^2 + y'^2 + z'^2\right)^{3/2}}(x', \gamma y', \gamma z').$$

The problem with this electric field in S is that it's still written in S' coordinates! Lorentz transforming to S coordinates (x, y, z):

$$\begin{aligned} x' &= \gamma(x - vt) \\ y' &= y \\ z' &= z. \end{aligned}$$

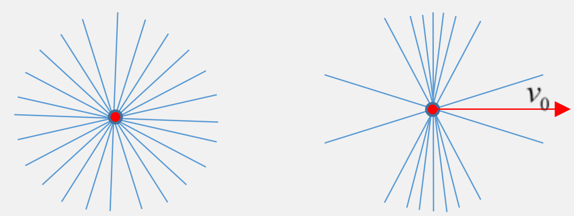
We now see that the electric field vector in the S frame points in the direction $\gamma(x - vt, y, z)$. But in this frame S at time t the charge q is at $(vt, 0, 0)$, so in fact the electric field at time t is pointing directly away from the position of the charge *at that instant*! (This should surprise you: remember that changes in an electric field caused by the source moving must propagate at the speed of light, not instantaneously: the space vector from the charge to the point where we're looking at the field is compressed in the x direction by γ^{-1}, but the perpendicular component of the field is *enhanced* by γ.) So we've found the direction of the electric field of a steadily moving charge, but what about the magnitude of the field? To find that, we need to put that denominator in the expression for $\vec{E}_S$ into S frame coordinates. Here it is:

$$\vec{E} = \frac{1}{4\pi\varepsilon_0}\frac{\gamma q}{\left(\gamma^2(x - vt)^2 + y^2 + z^2\right)^{3/2}}(x - vt, y, z).$$

Evidently, the field strength depends on direction. It's useful to write the position vector of the point P relative to the charge q in the S frame, $(x-vt, y, z)$, denoted by $\vec{R}$ in Griffiths, in spherical polar coordinates—all we really need is (R,θ) since the system has azimuthal symmetry about the direction of motion, and $x-vt = R\cos\theta,\ \sqrt{y^2+z^2} = R\sin\theta,$ so

$$\vec{E}_S = \frac{1}{4\pi\varepsilon_0}\frac{\gamma q\vec{R}}{\left(\gamma^2R^2\cos^2\theta + R^2\sin^2\theta\right)^{3/2}} = \frac{1}{4\pi\varepsilon_0}\frac{q\left(1-\beta^2\right)}{\left(1-\beta^2\sin^2\theta\right)^{3/2}}\cdot\frac{\vec{R}}{R^3}.$$

Notice that along the direction of motion, the electric field tends to zero as $\beta \to 1,\ v \to c,$ but in the direction perpendicular to the path the field increases without limit. (It's always radial.) *What about equipotentials*? Since they always cut field lines at right angles, they clearly would have to be circular? But the field strength varies with direction, so this gives a contradiction…? *Exercise*: figure this out. *Hint*: look up Coulomb gauge, Lorenz gauge...what are we using here?



Electric field lines of charge at rest

Electric field lines at relativistic speed

Electric and Magnetic Field Transformations

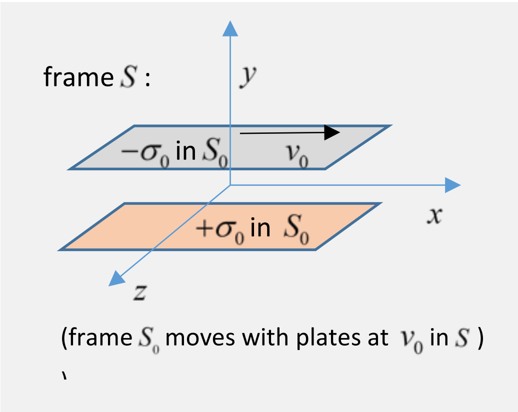
We showed at the beginning of this section that the electric field generated by uniformly charged plates in their rest frame, which in this section we'll label S_0, led to a field stronger by a factor γ_0 on measuring in a frame S, where frame S_0 is moving at velocity v_0 (parallel to the plates) relative to S. However, this is *not* the whole story—we looked at a special case, in which there was zero *magnetic* field in the first frame S_0. In the general case, both electric and magnetic fields in one frame contribute to electric and magnetic fields in another.

To find the appropriate equations, we need to have an initial frame with *both* electric and magnetic fields. It's not difficult: in the scenario we just discussed, there are both electric and magnetic fields in S, because the now moving charged plates are also current sheets, generating a magnetic field between them. So we just need to transform from this S to another frame $\bar{S}$, let's have $\bar{S}$ moving at v in the x direction relative to S.

The plates S_0, moving at v_0 in the x direction relative to S, constitute uniform current sheets, hence in S there is a magnetic field between them (see figure) in the z direction, $B_z = \mu_0\sigma v_0 = \mu_0\gamma_0\sigma_0 v_0$.

So here the only nonzero components of the electric and magnetic fields in S are E_y, B_z.

Now we're ready to transform to another frame $\bar{S}$ moving at v in the x direction relative to S.



To find how the fields transform, we just need to find the velocity $\bar{v}_0$ of the charged plates in $\bar{S}$, using the addition of velocities formula.

Since the plates are moving at v_0 in S, and $\bar{S}$ is moving at v in S, in $\bar{S}$ the plates are moving at

$$\bar{v}_0 = \frac{v_0 - v}{1 - v_0 v / c^2}.$$

Defining $\bar{\gamma}_0 = 1/\sqrt{1 - \bar{v}_0^2 / c^2}$, we have

$$\bar{\gamma}_0 = \frac{1}{\sqrt{1 - \left(\frac{v_0 - v}{1 - v_0 v / c^2}\right)^2}} = \frac{1 - v_0 v / c^2}{\sqrt{\left(1 - v_0 v / c^2\right)^2 - \left(v_0 - v\right)^2}} = \frac{1 - v_0 v / c^2}{\sqrt{1 - v_0^2 / c^2}\sqrt{1 - v^2 / c^2}}.$$

That is,

$$\bar{\gamma}_0 = \left(1 - v_0 v / c^2\right)\gamma_0 \gamma.$$

Hence the field transformation equations from E_y, B_z in S to $\bar{E}_y, \bar{B}_z$ in $\bar{S}$ are:

$$\begin{aligned} \bar{E}_y &= \bar{\gamma}_0 \frac{\sigma_0}{\varepsilon_0} = \left(1 - v_0 v / c^2\right)\gamma\gamma_0 \frac{\sigma_0}{\varepsilon_0} \\ &= \gamma\left(E_y - \frac{v}{c^2 \varepsilon_0 \mu_0} B_z\right) = \gamma\left(E_y - vB_z\right). \end{aligned}$$

(Recall $B_z = \mu_0 \sigma v_0 = \mu_0 \gamma \sigma_0 v_0$.) Here γ, v refer to the relative velocities of $S, \bar{S}$.
For the magnetic field,

$$\begin{aligned} \bar{B}_z &= \mu_0 \bar{\gamma} \sigma_0 \bar{v}_0 = \mu_0 \left(1 - v_0 v / c^2\right)\gamma\gamma_0 \sigma_o \frac{v_0 - v}{1 - v_0 v / c^2} \\ &= \mu_0 \gamma\gamma_0 \sigma_0 \left(v_0 - v\right) = \gamma\left(B_z - \left(v / c^2\right) E_z\right). \end{aligned}$$

Similarly, E_z, B_y transform as follows under an x direction boost:

$$\begin{aligned} \bar{E}_z &= \gamma\left(E_z + vB_y\right), \\ \bar{B}_y &= \gamma\left(B_y + \left(v / c^2\right) E_z\right). \end{aligned}$$

We've already seen that under this x direction boost

$$\bar{E}_x = E_x,$$

To complete the picture,

$$\bar{B}_x = B_x.$$

This is most simply proved by considering the magnetic field inside a long uniform solenoid, in a frame moving in the direction of its axis. Lorentz contraction will increase the number of windings per unit length, but time dilation will decrease the current by the same factor.

Important Special Cases:

If $\vec{B} = 0$ in S, then $\bar{\vec{B}} = -\left(1/c^2\right)\vec{v} \times \bar{\vec{E}}$,

If $\vec{E} = 0$ in S, then $\bar{\vec{E}} = \vec{v} \times \bar{\vec{B}}$.

In particular, the magnetic field of a charge in uniform motion is

$$\vec{B} = \frac{\mu_0}{4\pi} \frac{qv\left(1 - v^2/c^2\right)\sin\theta}{\left(1 - \left(v^2/c^2\right)\sin^2\theta\right)^{3/2}} \frac{\hat{\vec{\phi}}}{R^2}.$$

4/5/18

17. Lorentz Invariant Formulation of Electromagnetism

Contravariant and Covariant Four-Vectors

Recall we defined a contravariant four-vector as one that transformed under the Lorentz transformations in the same way as an incremental interval

$$dx^{\mu'} = \Lambda^{\mu'}{}_{\nu} dx^{\nu},$$

where for a boost $\beta = v/c$ along the x axis,

$$\Lambda^{\mu'}{}_{\nu} = \begin{pmatrix} \gamma & -\beta\gamma & 0 & 0 \\ -\beta\gamma & \gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$

Obviously (from the increment transformation equation above)

$$\Lambda^{\mu'}{}_{\nu} = \frac{\partial x^{\mu'}}{\partial x^{\nu}}.$$

Notice that

$$\Lambda_{\mu'}{}^{\nu} = \begin{pmatrix} \gamma & \beta\gamma & 0 & 0 \\ \beta\gamma & \gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix},$$

recalling that indices are lowered by multiplying by $g_{\mu\nu}$, raised with $g^{\mu\nu}$, both mean just multiplying the time-index term by -1. We see that

$$\Lambda^{\mu'}{}_{\nu} = \left(\Lambda_{\mu'}{}^{\nu}\right)^{-1}.$$

To formulate Maxwell's equations in a Lorentz invariant notation, we evidently need to know how the differential operators transform. In fact, it's simple -- it follows from the chain rule of differentiation:

$$\frac{\partial}{\partial x^{\mu'}} = \frac{\partial x^{\nu}}{\partial x^{\mu'}} \frac{\partial}{\partial x^{\nu}}.$$

Now,

$$\frac{\partial x^{\nu}}{\partial x^{\mu'}} \frac{\partial x^{\mu'}}{\partial x^{\rho}} = \frac{\partial x^{\nu}}{\partial x^{\rho}} = \delta^{\nu}_{\rho},$$

where δ^{ν}_{ρ} is the usual Kronecker delta in four dimensions, $\delta^{\nu}_{\rho} = 1$ if $\nu = \rho$, zero otherwise. That is,

$$\Lambda^{\mu'}{}_{\nu} = \frac{\partial x^{\mu'}}{\partial x^{\nu}}, \quad \left(\Lambda^{\mu'}{}_{\nu}\right)^{-1} = \Lambda_{\mu'}{}^{\nu} = \frac{\partial x^{\nu}}{\partial x^{\mu'}}.$$

Note that $dx^{\mu'} = \Lambda^{\mu'}{}_{\nu} dx^{\nu}$, $\quad \dfrac{\partial}{\partial x^{\mu'}} = \Lambda_{\mu'}{}^{\nu} \dfrac{\partial}{\partial x^{\nu}}$. This last equation is often written $\partial_{\mu'} = \Lambda_{\mu'}{}^{\nu} \partial_{\nu}$, to emphasize that the differential with respect to the contravariant vector is a covariant (subscript) vector.

Thus a product of covariant and a contravariant vector is invariant:

$$A^{\mu'} B_{\mu'} = \Lambda^{\mu'}{}_{\nu} \Lambda_{\mu'}{}^{\sigma} A^{\nu} B_{\sigma} = \delta^{\sigma}_{\nu} A^{\nu} B_{\sigma} = A^{\sigma} B_{\sigma}.$$

Note here that the two Λ's are inverses of each other, and so δ^{σ}_{ν} is the ordinary Kronecker δ, just 1's on the diagonal, zeroes elsewhere.

Four Current

An important example of a four-vector is the electric current,

$$J^{\mu} = \rho_0 U^{\mu}$$

where ρ_0 is the charge density in its rest frame, and U^{μ} is the local 4-velocity. The first component $\rho_0 U^0$ gives the local charge density (actually $c\gamma\rho_0$) in the (in general) moving frame, from the Lorentz volume contraction, the other three components are the three-dimensional current. (To get the signs right, recall $U^0 = dx^0 / d\tau = cdt / d\tau = c\gamma$.)

The experimentally established conservation of charge is written invariantly (remember $\partial_0 = \partial / \partial x^0 = (1/c)\partial / \partial t$) :

$$\frac{\partial J^{\mu}}{\partial x^{\mu}} = \partial_{\mu} J^{\mu} = 0.$$

This is really an "inner product" of the contravariant J^{μ} and the covariant ∂_{μ}.
(Just to make this clear: $\partial_{\mu} J^{\mu} = \partial_0 J^0 + \partial_1 J^1 + \partial_2 J^2 + \partial_3 J^3, \quad \partial_0 = -\partial^0, \quad \partial_1 = \partial^1 = \partial / \partial x,$ etc.)
There's another inner product we can make with ∂_{μ}, its product with itself:

$$\partial^{\mu}\partial_{\mu} = -\frac{1}{c^2}\frac{\partial^2}{\partial t^2} + \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}.$$

This will be invariant under a Lorentz transformation.
Recall now the wave equations for the vector and scalar potentials $\vec{A}, \varphi$:

$$\nabla^2\varphi - \frac{1}{c^2}\frac{\partial^2\varphi}{\partial t^2} = -\frac{\rho}{\varepsilon_0},$$
$$\nabla^2\vec{A} - \frac{1}{c^2}\frac{\partial^2\vec{A}}{\partial t^2} = -\mu_0\vec{J}.$$

If we divide every term in the first equation by c, the right hand sides of the two equations together form the four vector current:

$$-\mu_0\left(\frac{\rho}{\varepsilon_0\mu_0 c}, \vec{J}\right) = -\mu_0\left(c\rho, \vec{J}\right) = -\mu_0 J^{\mu}.$$

We can now rewrite the equations:

$$\partial^{\mu}\partial_{\mu}\left(\varphi / c, \vec{A}\right) = -\mu_0 J^{\mu}.$$

Remember these equations are nothing but a reformulation of Maxwell's equations, so must be true in all inertial frames. We know the right hand side is a four vector (it transforms appropriately) and the differential operator $\partial^\mu \partial_\mu$ is invariant, so, from the Lorentz invariance of the equations, it follows that

$$\left(\varphi / c, \vec{A}\right) = A^\mu$$

is a four vector.

There is one loose end here—the equations only have this simple "wave equation" form in the Lorenz gauge,

$$\vec{\nabla} \cdot \vec{A} = -\left(1/c^2\right) \partial \varphi / \partial t.$$

But this is just

$$\partial_\mu A^\mu = 0,$$

manifestly covariant, and the whole scheme is consistent.

Recall now that the magnetic field was generated from the vector potential by $\vec{B} = \vec{\nabla} \times \vec{A},$ or $B_i = \partial_j A_k - \partial_k A_j$ in an obvious notation. How does this work in four dimensions? To find out, we define

$$F^{\mu\nu} = \partial^\mu A^\nu - \partial^\nu A^\mu.$$

(This is the standard notation, with both indices up—so $\partial^\mu = \partial / \partial x_\mu,\ dx_\mu = g_{\mu\nu} dx^\nu$.)

This matrix is constructed from Lorentz vectors, so it will necessarily Lorentz transform as a tensor:

$$F^{\rho'\sigma'} = \Lambda^{\rho'}{}_{\mu} \Lambda^{\sigma'}{}_{\nu} F^{\mu\nu}.$$

Notice that it is an *antisymmetric* 4 x 4 matrix, so the four diagonal elements are zero, the other elements occur in pairs with opposite signs, and there are only six independent elements. And, we already know three of them:

$$\left(F^{23}, F^{31}, F^{12}\right) = \left(B_x, B_y, B_z\right).$$

But what about F^{01}, F^{02}, F^{03} ?

The first one

$$F^{01} = \partial^0 A^1 - \partial^1 A^0 = -\frac{1}{c}\frac{\partial A_x}{\partial t} - \frac{\partial}{\partial x}\frac{\varphi}{c},$$

remembering $\partial^0 = -\partial_0 = -\left(1/c\right)\partial / \partial t.$ This is nothing but the x component of

$$-\frac{1}{c}\left(\dot{\vec{A}}+\vec{\nabla}\varphi\right)=\frac{\vec{E}}{c},$$

so evidently

$$F^{\mu\nu}=\begin{pmatrix} 0 & E_x/c & E_y/c & E_z/c \\ -E_x/c & 0 & B_z & -B_y \\ -E_y/c & -B_z & 0 & B_x \\ -E_z/c & B_y & -B_x & 0 \end{pmatrix}.$$

Exercise: Use $F^{\bar{\rho}\bar{\sigma}}=\Lambda^{\bar{\rho}}{}_{\mu}\Lambda^{\bar{\sigma}}{}_{\nu}F^{\mu\nu}$ for the particular case of a boost $\beta=v/c$ along the x axis to find the Lorentz transformation equations for electric and magnetic fields. You should get the equations we derived by considering moving capacitors, etc. (To get you started, $\bar{F}^{01}=\Lambda^{0}{}_{\mu}\Lambda^{1}{}_{\nu}F^{\mu\nu}$ has only two nonzero terms.)

The Levi-Civita Tensor

An interesting higher order tensor (sometimes called the Levi-Civita tensor) is the set of numbers $\varepsilon^{\mu\nu\rho\sigma}$ defined as follows:

$$\varepsilon^{0123}=1.$$

For other indices, $\varepsilon^{\mu\nu\rho\sigma}=\pm 1$ if (μ,ν,ρ,σ) is $(0,1,2,3)$ in some order, and -1 for an odd permutation.

But if any of the four indices are equal, then $\varepsilon^{\mu\nu\rho\sigma}=0$. Thus, only 24 of the 256 elements are nonzero.

It turns out that this tensor is invariant under Lorentz transformations (although it changes sign under reflection, but that is not our concern here.)

To see how this comes about, look at the transformation equation:

$$\varepsilon^{\alpha'\beta'\gamma'\delta'}=\Lambda^{\alpha'}{}_{\mu}\Lambda^{\beta'}{}_{\nu}\Lambda^{\gamma'}{}_{\rho}\Lambda^{\delta'}{}_{\sigma}\varepsilon^{\mu\nu\rho\sigma}.$$

In fact, the expression on the right-hand side is the definition of the determinant $|\Lambda|$ if we take $(\alpha',\beta',\gamma',\delta')$ equal to $(0,1,2,3)$, and permutation is equivalent to permuting the rows of the matrix, which bring in the appropriate sign change in the determinant. If two are equal, that's the determinant of a matrix with two identical rows, which is zero (in finding the determinant, you can always subtract one row from another).

Actually this ε is sometimes called a pseudotensor, because it changes sign if the Lorentz transformation includes a reflection.

Exercise: how much of this is true in ordinary three dimensional space of the set ε_{ijk} ?

Dual Tensors

The Levi-Civita symbol is important in general relativity, its value here is that we can use it to generate another Lorentz invariant tensor from $F^{\mu\nu}$:

$$G^{\alpha\beta} = \tfrac{1}{2}\varepsilon^{\alpha\beta\gamma\delta}F_{\gamma\delta}.$$

This dual tensor is denoted by a script *F* in Jackson, and some books use $*F$, the so-called Hodge dual symbol. But the reason we mention this at all is that it illustrates the electric—magnetic duality nicely, and it makes some conserved quantities explicit, as we'll see below.

Compare $F^{\mu\nu}$ above with

$$G^{\mu\nu} = \begin{pmatrix} 0 & B_x & B_y & B_z \\ -B_x & 0 & -E_z/c & E_y/c \\ -B_y & E_z/c & 0 & -E_x/c \\ -B_z & -E_y/c & E_x/c & 0 \end{pmatrix}.$$

Just as with $F^{\mu\nu}$, the Lorentz transformation of $G^{\mu\nu}$ yields the equations we derived for transformation of electric and magnetic fields between frames.

We've already seen that the magnitude of a vector, say $A^{\mu}A_{\mu}$, is invariant under a Lorentz transformation. In the exact same way, we can show that $F^{\mu\nu}F_{\mu\nu}$ is invariant. Now

$$F^{\mu\nu}F_{\mu\nu} = F^{01}F_{01} + F^{02}F_{02} + F^{03}F_{03} + F^{12}F_{12} + F^{23}F_{23} + F^{31}F_{31},$$

Recalling that lowering a 0 suffix brings in a minus sign, we find that

$$-\frac{E^2}{c^2} + B^2 \text{ is Lorentz invariant.}$$

But what is the significance of this quantity? Unfortunately, we don't have time to pursue this here, but it's the Lagrangian density. (Recall that in classical mechanics the Lagrangian was the difference of kinetic energy and potential energy—something similar is happening here.)

Note this invariance tells us that if in some frame there is only a magnetic field, there is no frame in which there is only an electric field.

4/20/18

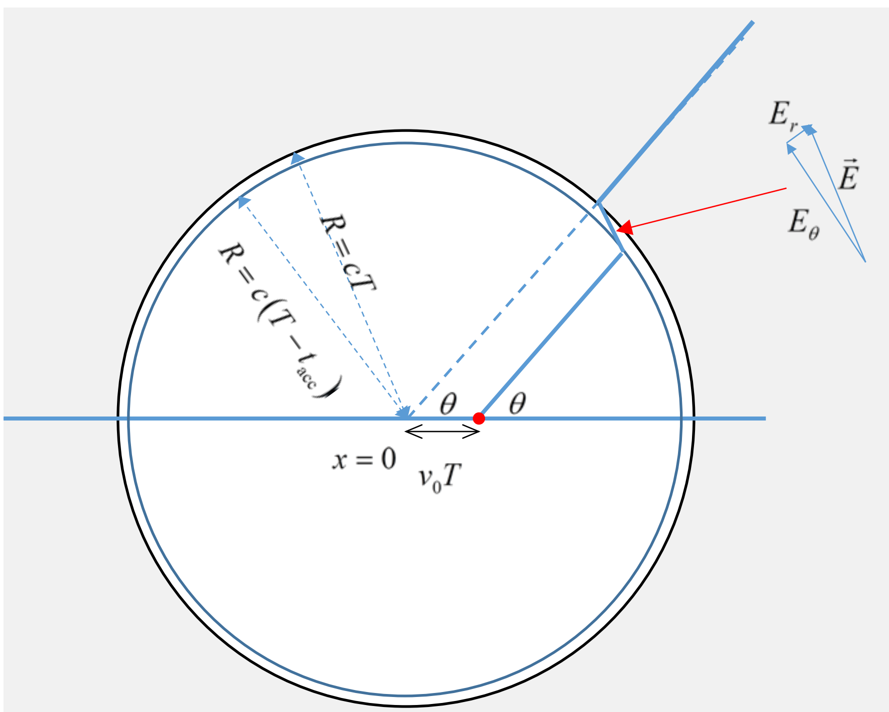
18. Radiation by an Accelerating Charge

Introduction

We'll begin with a simple derivation (due to Purcell) of the radiation from a nonrelativistic accelerating charge, and find a simple expression for the power. We'll then take the simplest possible generalization of this to the relativistic case, to find the central result, the radiation energy loss that determines the limits on accelerator energies for a given size machine. We'll contrast the results for linear and circular accelerators.
In the following lecture, we'll confirm the radiation energy result, deriving it from first principles following Lienard and Wiechert, then go on to find the angular distribution of the radiation, of central importance in the actual application of synchrotron radiation, for example.

Purcell's Derivation of the Larmor Formula

For the *nonrelativistic* case, here's a very elementary derivation due to Purcell. We take the electric field from a charge in steady motion to be the same as that from a charge at rest (so we're neglecting the previously discussed relativistic strengthening of the field in the perpendicular-to-motion direction.) Suppose, then, we have a charge q at rest at the origin for a long time, then, at $t = 0$, say, we apply a steady force F so the charge undergoes constant acceleration $a = \dot{v}$ for a *very* short time t_{acc} up to a speed $v_0 = at_{\text{acc}}$ at which point the force cuts out, and the charge from then on moves at this steady speed. During the acceleration, it will have moved a distance $\frac{1}{2}v_0 t_{\text{acc}}$ (too small to show on the accompanying diagram). Now consider what the electric field must look like at a much later time $T \gg t_{\text{acc}}$. At distances greater than cT from the original at rest position of the charge, the subsequent movement of the charge cannot have changed anything, so the electric field out there must still be radial pointing from the origin. But for distances less than $c(T - t_{\text{acc}})$, the



For $r > cT$, the electric field is radial from the origin, for $r < c(T - t_{\text{acc}})$ is radial from the charge.

(The distance $\frac{1}{2}v_0 t_{\text{acc}}$ is too small to show.)

radial field lines must emanate from the moving charge. (Recall that we've already established that the field lines from a charge in steady motion are radial from the instantaneous position of the charge: and at distances $r < c\left(T - t_{\text{acc}}\right)$, the field can only depend on the history of the charge after it stopped accelerating.)

So there must be a thin shell, $c\left(T - t_{\text{acc}}\right) < R < cT$, in which the electric field lines connect. (The amount of electric field "flowing out" in the cone of angle θ must be the same before and after, and the electric field lines are continuous at all times.) $\vec{E}$ must zigzag as shown, and we see that in the limit of large R,

$$\frac{E_\theta}{E_r} = \frac{v_0 T \sin\theta}{c t_{\text{acc}}} = \frac{v_0 R \sin\theta}{c^2 t_{\text{acc}}}.$$

Now, E_r must be essentially the same inside the shell as outside (Gauss' law),

$$E_r = \frac{1}{4\pi\varepsilon_0}\frac{q}{R^2}.$$

Hence,

$$E_\theta = \frac{v_0 R \sin\theta}{c^2 t_{\text{acc}}} \cdot \frac{1}{4\pi\varepsilon_0}\frac{q}{R^2} = \frac{1}{4\pi\varepsilon_0}\frac{v_0 \sin\theta}{c^2 t_{\text{acc}}} \cdot \frac{q}{R}.$$

That is to say, E_θ goes down with distance only as $1/R.$ Evidently, this is an outgoing *electromagnetic wave* -- the same energy propagates outwards independent of the size sphere we take.

We'll take the outgoing radiation to be Pt_{acc}, the radiation is uniform over a time t_{acc}, so P is the power. We simply need to find the energy in the electric field in the shell between $T - t_{\text{acc}}$ and T, then remember to double it, because an electromagnetic wave has equal energies in the electric and magnetic fields.

Given the volume of the shell is $4\pi R^2 c t_{\text{acc}}$, and $\overline{\sin^2\theta} = \tfrac{2}{3}$, and putting in $\tfrac{1}{2}$ because the electric field is only half the wave energy,

$$\begin{aligned}
\tfrac{1}{2} P t_{\text{acc}} &= \int_{\text{shell}} \tfrac{1}{2}\varepsilon_0 E^2 dV \\
&= \tfrac{1}{2}\varepsilon_0 \cdot 4\pi R^2 c t_{\text{acc}} \cdot \frac{\tfrac{2}{3}}{\left(4\pi\varepsilon_0\right)^2} \cdot \left(\frac{v_0 R}{c^2 t_{\text{acc}}}\right)^2 \cdot \left(\frac{q}{R^2}\right)^2 \\
&= \frac{\tfrac{2}{3}}{4\pi\varepsilon_0} \cdot \frac{v_0^2}{c^3 t_{\text{acc}}} \cdot q^2.
\end{aligned}$$

Now we put $v_0 = a t_{\text{acc}}$ to find

$$P = \frac{1}{4\pi\varepsilon_0} \cdot \frac{2}{3} \cdot \frac{a^2 q^2}{c^3} = \frac{1}{6\pi\varepsilon_0} \cdot \frac{a^2 q^2}{c^3} = \frac{\mu_0}{6\pi} \cdot \frac{a^2 q^2}{c}.$$

This is called the *Larmor formula*.
Notice that the power depends on the *square* of the acceleration, so acceleration in the opposite direction—deceleration—would emit the same power. Furthermore, the radiation distribution is obviously azimuthally symmetric about the direction of the acceleration, so the radiated total momentum must be zero. It follows that on going to a different Lorentz frame, the energy in this shell of radiation increases by a factor γ, as does the time t_{acc} over which it's emitted, so the *power* is actually unchanged.

Relativistic Generalization of the Larmor Formula

It turns out that the nonrelativistic Larmor formula is not difficult to write as the limit of a relativistic scalar. Recall that the relativistic four-velocity, $U^\mu = dx^\mu / d\tau$, goes to $(c, \vec{v})$ in the nonrelativistic limit, and the four-acceleration $a^\mu = dU^\mu / d\tau = d^2 x^\mu / d\tau^2 \to (0, \vec{a})$, so the invariant scalar $a^\mu a_\mu \to \vec{a}^2$.
Hence the obvious generalization to the relativistic case is

$$P = \frac{\mu_0}{6\pi} \cdot \frac{a^\mu a_\mu q^2}{c}.$$

This is not a rigorous derivation, there could be other terms in the relativistic expression that go to zero in the nonrelativistic limit. But from the form of the Lienard Weichert fields, which we cover in the next lecture, the velocity dependence can only be on $\vec{\beta}, \dot{\vec{\beta}}$ (no higher derivatives) and given this constraint, according to Jackson, this is the only possible relativistic generalization of Larmor's formula.
However, for it to be useful, we need to translate this invariant four-acceleration magnitude into a more useable form, that is, in terms of ordinary (lab frame) velocities and accelerations, going from proper time to lab time, using

$$\frac{d}{d\tau} = \frac{1}{\sqrt{1-\beta^2}} \frac{d}{dt}.$$

Transforming to lab time t, then, and we'll sometimes use a dot to denote differentiation with respect to t,

$$a^\mu = \frac{d^2 x^\mu}{d\tau^2} = \frac{1}{\sqrt{1-\beta^2}} \frac{d}{dt} \frac{1}{\sqrt{1-\beta^2}} \frac{dx^\mu}{dt},$$

and noting that

$$\dot{\gamma} = \frac{d}{dt}\frac{1}{\sqrt{1-\beta^2}} = \frac{\vec{\beta}\cdot\dot{\vec{\beta}}}{\left(1-\beta^2\right)^{3/2}} = \gamma^3\vec{\beta}\cdot\dot{\vec{\beta}},$$

we have

$$\begin{aligned} a^\mu &= \left(a^0, a^i\right) \\ &= \left(\frac{1}{\sqrt{1-\beta^2}}\frac{d}{dt}\frac{1}{\sqrt{1-\beta^2}}\frac{d(ct)}{dt}, \frac{1}{\sqrt{1-\beta^2}}\frac{d}{dt}\frac{1}{\sqrt{1-\beta^2}}\frac{dx^i}{dt}\right) \\ &= c\left(\gamma\dot{\gamma}, \gamma\dot{\gamma}\beta^i + \gamma^2\dot{\beta}^i\right), \end{aligned}$$

Then, using (from the previous equation) $\dot{\gamma} = \gamma^3\vec{\beta}\cdot\dot{\vec{\beta}}$ to eliminate $\dot{\gamma}$, we find

$$a^\mu = \left(a^0, a^i\right) = c\gamma^2\left(\gamma^2\left(\beta\cdot\dot{\beta}\right), \gamma^2\beta^i\left(\beta\cdot\dot{\beta}\right) + \dot{\beta}^i\right).$$

Hence

$$\begin{aligned} \left(a^\mu a_\mu\right)/c^2\gamma^4 &= -\gamma^4\left(\beta\cdot\dot{\beta}\right)^2 + \gamma^4\beta^2\left(\beta\cdot\dot{\beta}\right)^2 + 2\gamma^2\left(\beta\cdot\dot{\beta}\right)^2 + \dot{\beta}^2 \\ &= -\gamma^4\left(1-\beta^2\right)\left(\beta\cdot\dot{\beta}\right)^2 + 2\gamma^2\left(\beta\cdot\dot{\beta}\right)^2 + \dot{\beta}^2 \\ &= -\gamma^2\left(\beta\cdot\dot{\beta}\right)^2 + 2\gamma^2\left(\beta\cdot\dot{\beta}\right)^2 + \dot{\beta}^2 \\ &= \gamma^2\left[\left(1-\beta^2\right)\dot{\beta}^2 + \left(\beta\cdot\dot{\beta}\right)^2\right] \\ &= \gamma^2\left[\dot{\beta}^2 - \left(\beta\times\dot{\beta}\right)^2\right]. \end{aligned}$$

Hence

$$P = \frac{\mu_0 c q^2\gamma^6}{6\pi}\left[\dot{\beta}^2 - \left(\beta\times\dot{\beta}\right)^2\right] = \frac{2}{3}\cdot\frac{1}{4\pi\varepsilon_0}\cdot\frac{q^2\gamma^6}{c}\cdot\left[\dot{\beta}^2 - \left(\beta\times\dot{\beta}\right)^2\right].$$

This is the famous result of Lienard and Weichert, presented by Jackson in terms of the momentum of a charged particle:

$$P = \frac{2}{3}\frac{1}{4\pi\varepsilon_0}\frac{q^2}{m^2c^3}\left(\frac{dp_\mu}{d\tau}\cdot\frac{dp^\mu}{d\tau}\right).$$

Jackson remarks that the appearance of mass in the denominator here means radiation power loss is far more serious for electrons than for protons, which is of course true, but the radiation doesn't depend on

the mass of the particle, this is not gravitational radiation, the point is that *for a given speed and acceleration* the radiation from an electron is the same as for a proton.
That being said, the energy of a particle $E=\gamma mc^2$, so far an electron and a proton of equal energy E, we have $\gamma_e m_e = \gamma_p m_p$, so the energy loss for the electron, notice the γ^6, is far greater. (As we'll soon see, it drops to γ^4 in circular motion, but that's still very large.)

Linear Accelerator

For a particle traveling along a straight line and accelerating, $\vec{\beta}\times\dot{\vec{\beta}}=0$ so the rate of energy radiation

$$\begin{aligned} P &= \frac{2}{3}\cdot\frac{1}{4\pi\varepsilon_0}\cdot\frac{q^2\gamma^6}{c}\cdot\dot{\beta}^2 \\ &= \frac{2}{3}\cdot\frac{1}{4\pi\varepsilon_0}\cdot\frac{q^2}{m^2c^3}\left(\frac{dp}{dt}\right)^2. \end{aligned}$$

(That last line, connecting with Jackson, uses $\dot{\gamma}=\gamma^3\vec{\beta}\cdot\dot{\vec{\beta}}=\gamma^3\beta\dot{\beta}$ and

$$\dot{p}=\dot{\gamma}mc\beta+\gamma mc\dot{\beta}=\gamma^3 mc\dot{\beta}\beta^2+\gamma^3\left(1-\beta^2\right)mc\dot{\beta}=\gamma^3 mc\dot{\beta}.)$$

Using that the accelerating force $\left(dp/dt\right)=\left(dE/dx\right)$,

$$P=\frac{2}{3}\frac{1}{4\pi\varepsilon_0}\frac{q^2}{m^2c^3}\left(\frac{dE}{dx}\right)^2.$$

This means the rate of radiative energy loss in linear motion depends only on the external force that determines the change of particle energy with distance—not on the particle's actual energy.

So the ratio of radiative energy loss to power being supplied is

$$\frac{P}{\left(dE/dt\right)}=\frac{2}{3}\frac{1}{4\pi\varepsilon_0}\frac{q^2}{m^2c^3}\frac{1}{v}\frac{dE}{dx}\xrightarrow[v\to c]{}\frac{2}{3}\frac{1}{4\pi\varepsilon_0}\frac{\left(q^2/mc^2\right)}{mc^2}\frac{dE}{dx}.$$

For an electron, the length $\frac{1}{4\pi\varepsilon_0}\frac{q^2}{mc^2}$ is called the classical radius (for the charge confined to this radius, the electric field energy is the rest mass) and is of order 10^{-15} meters. In a real linear accelerator, essentially no energy is gained in that distance, so radiative energy loss is not a problem.

Circular Accelerator

This time $\left(\beta\times\dot{\beta}\right)^2=\beta^2\dot{\beta}^2$ so

$$P=\frac{2}{3}\cdot\frac{1}{4\pi\varepsilon_0}\cdot\frac{q^2\gamma^6}{c}\left[\dot{\beta}^2-\left(\beta\times\dot{\beta}\right)^2\right]=\frac{2}{3}\cdot\frac{1}{4\pi\varepsilon_0}\cdot\frac{q^2\gamma^4}{c}\dot{\beta}^2,$$

and in terms of momentum, since $\dot{p}=\gamma mc\dot{\beta}$ for sideways acceleration,

$$P=\frac{2}{3}\cdot\frac{1}{4\pi\varepsilon_0}\cdot\frac{q^2\gamma^2}{m^2c^3}\left(\frac{dp}{dt}\right)^2,$$

so for going in a circle of radius r, for which $\dot{\beta}=c\beta^2/r$,

$$P=\frac{2}{3}\cdot\frac{cq^2\gamma^4}{4\pi\varepsilon_0}\cdot\frac{\beta^4}{r^2}.$$

Note that this doesn't depend on the mass of the particle—of course—it's from the accelerating charge. But the total energy is $E=\gamma mc^2$, so the power loss is far more serious for electrons than for protons.

The energy loss per revolution is

$$\delta E=\frac{2\pi r}{c\beta}P=\frac{2\pi r}{c\beta}\frac{2}{3}\cdot\frac{cq^2\gamma^4}{4\pi\varepsilon_0}\cdot\frac{\beta^4}{r^2}=\frac{1}{3\varepsilon_0}\cdot\frac{\gamma^4\beta^4}{r}.$$

and in numbers (from Jackson) $\delta E\left(\mathrm{MeV}\right)=8.85\times10^{-2}\frac{\left[E\left(\mathrm{GeV}\right)\right]^4}{r\left(\mathrm{meters}\right)}.$

For the Cornell electron synchrotron, a 10 GeV machine with a radius of 100 meters, loss per turn at maximum design energy is 8.85 MeV, energy supplies is 10 MeV.

The inner product $G^{\alpha\beta}G_{\alpha\beta}$ gives the same invariant, by inspection, but $F^{\alpha\beta}G_{\alpha\beta}=\vec{E}\cdot\vec{B}$, evidently another Lorentz invariant.

Exercise: check that Maxwell's equations can be written $\partial_\nu F^{\mu\nu}=\mu_0 J^\mu$, $\partial_\nu G^{\mu\nu}=0$ and that the Lorentz force law is the nonrelativistic limit of $dp^\mu/d\tau=qU_\nu F^{\mu\nu}$.

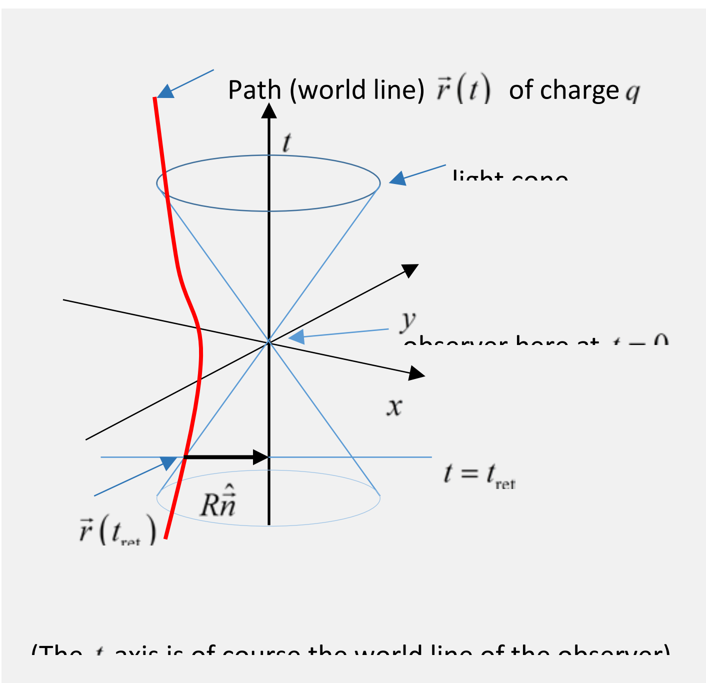
19. The Lienard-Weichert Potentials and Fields

Potentials

The correct expressions for the potentials φ and $\vec{A}$ were found long ago, before relativity, by Lienard and Wiechert. They are:

$$\varphi(\vec{x},t)=\frac{1}{4\pi\varepsilon_0}\left[\frac{q}{R\left(1-\vec{\beta}\cdot\vec{n}\right)}\right]_{\text{ret}}, \quad \vec{A}=\frac{\mu_0 c}{4\pi}\left[\frac{q\vec{\beta}}{R\left(1-\vec{\beta}\cdot\vec{n}\right)}\right]_{\text{ret}}.$$

The bracket $[\;]_{\text{ret}}$ means that the contents must be evaluated at the retarded time, meaning the potential is determined by the position (and velocity) of the charge at time earlier by t_{ret} such that $ct_{\text{ret}}=R,$ the distance of the charge at that instant from the point of observation. In other words, information about the charge position encoded in the potential is transmitted at the speed of light.



In this diagram, we take the origin to be at the point and time of the observation, to simplify the picture. The field observed at $\vec{r}=0, t=0$ arises from the point in spacetime where the world path of the charge intersected the backward light cone from the point of observation.

This naturally gives rise to the expressions above, except for that extra factor $1-\vec{\beta}\cdot\hat{\vec{n}}$. Where did that come from?

It arises because we must integrate over all past times, using a delta function to pick out the right one:

$$\varphi(\vec{x},t)=\frac{1}{4\pi\varepsilon_0}\int\limits_{t'<t}\frac{q}{R}\delta\left(t'-R(t')/c\right)dt'.$$

From the well-known formula

$$\int g(t)\delta(f(t))dt = g(t_0)/|f'(t_0)|$$

(assuming a single zero of $f(t)$ at t_0), we find

$$\varphi(\vec{x},t) = \frac{1}{4\pi\varepsilon_0}\frac{q}{R_{\text{ret}}}\frac{1}{\left(1-\vec{\beta}_{\text{ret}}\cdot\hat{\vec{n}}\right)},$$

because the term multiplying $\dot{R}(t_{\text{ret}})$ that comes from differentiating the delta function is just the component of the velocity in the direction $\hat{\vec{n}}$ from the charge at $\vec{r}(t_{\text{ret}})$ to the point $\vec{x}=0$ at which we are measuring the field.

This is rather a formal derivation: is there some more intuitive way to see where the factor comes from? Try this. As we discussed long ago in talking about Green's functions, the charge is a source, from which the field emanates. Imagine it to be constantly transmitting little packets of information, proportional to its strength. We detect the field at some other point from the rate at which these packets arrive. If we're moving towards the source, they come in that much faster, by exactly this term. This is just the same as the Doppler effect. (This argument is not invoking time dilation, relativity, etc.: Lienard and Wiechert wrote it long before Einstein!)

The Electric and Magnetic Fields

Having found the potentials, all we need do to evaluate the fields is to differentiate:

$$\vec{B} = \vec{\nabla}\times\vec{A}, \quad \vec{E} = -\vec{\nabla}\varphi - \partial\vec{A}/\partial t$$

But this isn't so easy— $\vec{\nabla}\times\vec{A}$ means differentiating $\vec{A}$ with respect to $\vec{x}$ at fixed t. Now the values of $\vec{A}$ at two neighboring points $\vec{x}$ and $\vec{x}+\Delta\vec{x}$ at time t correspond to different times t_{ret}. This will affect $R\hat{\vec{n}}$ and must be included in differentiating the potentials.

Let's start by considering R, the distance from $\vec{r}(t_{\text{ret}})$ to $\vec{x}$. We take it that the path the particle follows, $\vec{r}(t_{\text{ret}})$, has been set in advance. This means that R can be thought of as a function of $\vec{x}$ and t_{ret} *only*, its dependence on t_{ret} being given by the known function $\vec{r}(t_{\text{ret}})$. The velocity of the particle at time t_{ret} is obviously $\partial\vec{r}(t_{\text{ret}})/\partial t_{\text{ret}}$. (Since t_{ret} is defined as a function of t from $R = |\vec{x}-\vec{r}(t_{\text{ret}})| = c(t-t_{\text{ret}})$, this may seem confusing. Why isn't the velocity just $\partial\vec{r}(t_{\text{ret}})/\partial t$? The point is that $\vec{r}(t_{\text{ret}})$ is the position at the earlier time t_{ret}, and if the particle is moving at constant velocity $\vec{v}$, for example, then when one second of the "earlier time" has elapsed, the particle will have moved $\vec{v}$ meters. For such a particle, the path is $\vec{r}\left(t_{\text{ret}}\right)=\vec{v}t_{\text{ret}}$. Think of a car going along a road at 60 mph, passing milestones each of which has a clock. As it passes each clock, the clock will read exactly one minute later than the previous clock did as the car passed. Now imagine someone observing this journey with a telescope from far away. That observer will also see the milestone clocks each reading one minute later as the car passes—and each such observation is of a retarded position at the corresponding retarded time.)

For fixed $\vec{x}$,

$$\partial R / \partial t_{\text{ret}} = -\hat{\vec{n}}\cdot\vec{v}$$

because $R\hat{\vec{n}}$ points from $\vec{r}\left(t_{\text{ret}}\right)$ to $\vec{x}$, so if $\vec{r}\left(t_{\text{ret}}\right)$ is moving at $\vec{v}$, $R\hat{\vec{n}}$ is being eaten up from the back at the rate given by the right-hand side of the above equation, that is, the component of $\vec{r}\left(t_{\text{ret}}\right)$'s velocity along the line of the vector.

N0w from $R=c\left(t-t_{\text{ret}}\right)$, differentiating R with respect to time at fixed $\vec{x}$,

$$\frac{\partial R}{\partial t}=c\left(1-\frac{\partial t_{\text{ret}}}{\partial t}\right)=\frac{\partial R}{\partial t_{\text{ret}}}\frac{\partial t_{\text{ret}}}{\partial t}=-\hat{\vec{n}}\cdot\vec{v}\,\frac{\partial t_{\text{ret}}}{\partial t}$$

from this we find

$$\frac{\partial t_{\text{ret}}}{\partial t}=\frac{1}{1-\hat{\vec{n}}\cdot\vec{\beta}}\,,\quad \vec{\beta}=\vec{v}/c.$$

Or

$$\frac{\partial}{\partial t}=\left(1-\hat{\vec{n}}\cdot\vec{\beta}\right)\frac{\partial}{\partial t_{\text{ret}}}$$

both partial derivatives being understood to be at fixed $\vec{x}$.

(Actually this is easy to see. Consider the one-dimensional case: $\vec{r}\left(t_{\text{ret}}\right)$ is moving along the *x*-axis towards $\vec{x}$ (which is itself on the axis) at steady speed v. If Δt_{ret} is one second, what is Δt? If at $t_{\text{ret}}=0$ the distance from $\vec{r}\left(t_{\text{ret}}\right)$ to $\vec{x}$ is R, then $t=R/c$. When $t_{\text{ret}}=1$ second, $\vec{r}\left(t_{\text{ret}}\right)$ has moved v meters towards $\vec{x}$, and the corresponding t is now $t=1+\left(R-v\right)/c$. Hence $\Delta t_{\text{ret}}/\Delta t=1/\left(1-v/c\right)$.)

Now for the grad operator. The grad operator takes derivatives at constant t by definition, so since $R = c\left(t - t_{\text{ret}}\right)$,

$$\vec{\nabla} R = -c\vec{\nabla} t_{\text{ret}}.$$

But we can also approach $\vec{\nabla} R$ from $R = \left|\vec{x} - \vec{r}\left(t_{\text{ret}}\right)\right|$. We must first differentiate with respect to $\vec{x}$ at constant t_{ret}, then add the contribution from the variation of $\vec{r}\left(t_{\text{ret}}\right)$, but that latter is just the contribution from varying t_{ret} at fixed $\vec{x}$.

Thus
$$\vec{\nabla} R = \vec{\nabla}_{\vec{x}} R + \frac{\partial R}{\partial t_{\text{ret}}} \vec{\nabla} t_{\text{ret}} = \hat{\vec{n}} - \left(\hat{\vec{n}} \cdot \vec{v}\right) \vec{\nabla} t_{\text{ret}}.$$

Putting this together with $\vec{\nabla} R = -c\vec{\nabla} t_{\text{ret}}$, we see that

$$\vec{\nabla} t_{\text{ret}} = -\frac{\hat{\vec{n}}}{c - \hat{\vec{n}} \cdot \vec{v}} = -\frac{\hat{\vec{n}}}{c\left(1 - \hat{\vec{n}} \cdot \vec{\beta}\right)}.$$

(This is also easy to see—consider first the situation where the charge is stationary at a fixed point $\vec{r}\left(t_{\text{ret}}\right)$. Then $\vec{\nabla} t_{\text{ret}}$ (meaning of course $\vec{\nabla}_{\vec{x}} t_{\text{ret}}$) is in the direction $-R\hat{\vec{n}}$, since perpendicular increments correspond to the same time, visualizing a sphere of constant t_{ret} centered at $\vec{r}\left(t_{\text{ret}}\right)$, and the sign is negative because points further out in the $R\hat{\vec{n}}$ direction at fixed t correspond to earlier t_{ret}. And, its value is obviously $1/c$. If now $\vec{r}\left(t_{\text{ret}}\right)$ varies with t_{ret}, the only relevant variation is in the direction $\hat{\vec{n}}$, giving the value above.)

It follows that

$$\vec{\nabla} = \vec{\nabla}_{\vec{x}} + \left(\vec{\nabla} t_{\text{ret}}\right) \frac{\partial}{\partial t_{\text{ret}}} = \vec{\nabla}_{\vec{x}} - \frac{\hat{\vec{n}}}{c\left(1 - \vec{\beta} \cdot \hat{\vec{n}}\right)} \frac{\partial}{\partial t_{\text{ret}}}$$

We are now have all the information we need to calculate the fields from the potentials. The electric field is given by $\vec{E} = -\vec{\nabla}\varphi - \partial\vec{A}/\partial t$, and putting in the expressions we found above for the derivatives it is a straightforward calculation. However, it is also fairly lengthy, and not particularly illuminating, so we put it in an appendix.

The final expression for the electric field is:

$$\vec{E}=\frac{q}{4\pi\varepsilon_0 R^2}\left[\frac{\hat{\vec{n}}-\vec{\beta}}{\gamma^2\left(1-\vec{\beta}\cdot\hat{\vec{n}}\right)^3}\right]_{\rm ret}+\frac{q}{4\pi\varepsilon_0 cR}\left[\frac{\hat{\vec{n}}\times\left[\left(\hat{\vec{n}}-\vec{\beta}\right)\times\dot{\vec{\beta}}\right]}{\left(1-\vec{\beta}\cdot\hat{\vec{n}}\right)^3}\right]_{\rm ret}.$$

Notice the first term doesn't depend on the acceleration, just the velocity, so it can't be a radiation field—in any case, it goes down as the inverse square, so it's just the Coulomb field we found previously for a steadily moving charge. The direction of the field, as we discussed previously, is to the point where the charge would be at the present instant if it were traveling at a constant velocity, that is the direction $\hat{\vec{n}}-\vec{\beta},$ with $\hat{\vec{n}}$ being the unit vector to the retarded position.

The second term is the radiation: it goes as $1/R,$ and the magnetic field is perpendicular to the electric and of magnitude $E/c.$

Angular Distribution of Radiation

Linear Acceleration

The outward energy flow can be found from the radial component of the Poynting vector $\vec{S}=\frac{1}{\mu_0}\vec{E}\times\vec{B},$ using the expression just derived for the radiation electric and magnetic fields:

$$\left[\vec{S}\cdot\hat{\vec{n}}\right]_{\rm ret}=\frac{1}{\mu_0}\left(\frac{q}{4\pi\varepsilon_0 cR}\right)^2\frac{1}{c}\left[\left|\frac{\hat{\vec{n}}\times\left[\left(\hat{\vec{n}}-\vec{\beta}\right)\times\dot{\vec{\beta}}\right]}{\left(1-\vec{\beta}\cdot\hat{\vec{n}}\right)^3}\right|^2\right]_{\rm ret}.$$

Looking first at the nonrelativistic case,

$$\left[\vec{S}\cdot\hat{\vec{n}}\right]_{\rm ret}=\frac{1}{\mu_0}\frac{q^2}{16\pi^2\varepsilon_0^2c^3R^2}\left|\hat{\vec{n}}\times\left[\hat{\vec{n}}\times\dot{\vec{\beta}}\right]\right|^2=\frac{q^2}{16\pi^2\varepsilon_0 cR^2}.\left|\hat{\vec{n}}\times\left[\hat{\vec{n}}\times\dot{\vec{\beta}}\right]\right|^2=\frac{\mu_0 q^2}{16\pi^2 cR^2}.\left|\hat{\vec{n}}\times\left[\hat{\vec{n}}\times\dot{\vec{v}}\right]\right|^2,$$

so
$$\frac{dP}{d\Omega}=\frac{\mu_0 q^2}{16\pi^2 c}\left|\dot{\vec{v}}\right|^2\sin^2\theta.$$

Actually, we've already seen this result: it's just dipole radiation (see the earlier lecture: a radiating dipole can be thought of as one fixed charge, one accelerating).

Going now to highly relativistic accelerating particles, the denominator dominates: the radiation is far greater near (but not actually in) the forward direction.

The above general expression is for radiation detected at time $t,$ remember $t'=t-R(t')/c,$ to get the radiation emitted by the particle over some interval dt' we must include a factor derived earlier, $dt/dt'=\left(1-\vec{\beta}\cdot\hat{\vec{n}}\right),$

$$\frac{dP(t')}{d\Omega} = \frac{\mu_0 q^2}{16\pi^2 c} \frac{\left|\hat{\vec{n}} \times \left[\left(\hat{\vec{n}} - \vec{\beta}\right) \times \dot{\vec{\beta}}\right]\right|^2}{\left(1 - \vec{\beta} \cdot \hat{\vec{n}}\right)^5}$$

and for linear motion

$$\frac{dP(t')}{d\Omega} = \frac{\mu_0 q^2 \dot{v}^2}{16\pi^2 c} \frac{\sin^2\theta}{(1-\beta\cos\theta)^5}.$$

For β close to 1, almost all the radiation is in the forward direction.

Writing $\gamma^{-2} = \left(1-\beta^2\right) \cong 2(1-\beta)$ we write the denominator $1-\beta\cos\theta \cong \frac{1}{2}\gamma^{-2} + \frac{1}{2}\theta^2$ to get for small angles

$$\frac{dP(t')}{d\Omega} = \frac{2\mu_0 q^2 \dot{v}^2}{\pi^2 c} \gamma^8 \frac{(\gamma\theta)^2}{\left(1+\gamma^2\theta^2\right)^5},$$

peaking at $\theta_{\max} \simeq 1/2\gamma$.

Circular Acceleration

Following Jackson, we take $\vec{\beta} = \beta\hat{z}, \quad \dot{\vec{\beta}} = \dot{\beta}\hat{x}, \quad \hat{n} = \sin\theta\cos\phi\,\hat{x} + \sin\theta\sin\phi\,\hat{y} + \cos\theta\,\hat{z}$ in

$$\frac{dP(t')}{d\Omega} = \frac{\mu_0 q^2}{16\pi^2 c} \frac{\left|\hat{\vec{n}} \times \left[\left(\hat{\vec{n}} - \vec{\beta}\right) \times \dot{\vec{\beta}}\right]\right|^2}{\left(1 - \vec{\beta} \cdot \hat{\vec{n}}\right)^5}$$

to find with lots of algebra

$$\frac{dP(t')}{d\Omega} = \frac{\mu_0 q^2 \dot{v}^2}{16\pi^2 c} \cdot \frac{1}{(1-\beta\cos\theta)^3} \cdot \left[1 - \frac{\sin^2\theta\cos^2\phi}{\gamma^2(1-\beta\cos\theta)^2}\right].$$

Appendix

Detailed derivation of the fields from the potentials, following Panovsky and Phillips.

The electric field is given by $\vec{E} = -\vec{\nabla}\varphi - \partial\vec{A}/\partial t$.

Putting in the Lienard-Wiechert potentials for the point charge,

$$\varphi(\vec{x},t)=\frac{1}{4\pi\varepsilon_0}\left[\frac{q}{R\left(1-\vec{\beta}\cdot\vec{n}\right)}\right]_{\text{ret}}, \quad \vec{A}=\frac{\mu_0 c}{4\pi}\left[\frac{q\vec{\beta}}{R\left(1-\vec{\beta}\cdot\vec{n}\right)}\right]_{\text{ret}},$$

we find

$$\vec{E}=\frac{q}{4\pi\varepsilon_0}\left(-\vec{\nabla}\frac{1}{R\left(1-\vec{\beta}\cdot\hat{\vec{n}}\right)}-\frac{\partial}{\partial t}\frac{\vec{v}}{R\left(1-\vec{\beta}\cdot\hat{\vec{n}}\right)c^2}\right)$$

To evaluate this expression, we simply substitute the operators:

$$\frac{\partial}{\partial t}=\left(1-\hat{\vec{n}}\cdot\vec{\beta}\right)\frac{\partial}{\partial t_{\text{ret}}}, \quad \vec{\nabla}=\vec{\nabla}_{\vec{x}}-\frac{\hat{\vec{n}}}{c\left(1-\vec{\beta}\cdot\hat{\vec{n}}\right)}\frac{\partial}{\partial t_{\text{ret}}} .$$

To make the expressions less cumbersome, we write (following Panovsky and Phillips):

$$R\left(1-\vec{\beta}\cdot\hat{\vec{n}}\right)=s.$$

In this notation,

$$\frac{\partial}{\partial t}=\frac{s}{R}\frac{\partial}{\partial t_{\text{ret}}}, \quad \vec{\nabla}=\vec{\nabla}_{\vec{x}}-\frac{R\hat{\vec{n}}}{sc}\frac{\partial}{\partial t_{\text{ret}}}$$

and

$$\vec{E}=\frac{q}{4\pi\varepsilon_0}\left(-\vec{\nabla}\frac{1}{s}-\frac{\partial}{\partial t}\frac{\vec{v}}{sc^2}\right).$$

Combining the above two equations,

$$\vec{E}=\frac{q}{4\pi\varepsilon_0}\left[\frac{1}{s^2}\nabla_{\vec{x}}s-\frac{R\hat{\vec{n}}}{cs^3}\frac{\partial s}{\partial t_{\text{ret}}}-\frac{R}{s^2c}\dot{\vec{\beta}}+\frac{R\vec{\beta}}{cs^3}\frac{\partial s}{\partial t_{\text{ret}}}\right]$$

Now $s=R\left(1-\vec{\beta}\cdot\hat{\vec{n}}\right)=R-\vec{\beta}\cdot\vec{R}$, so $\nabla_{\vec{x}}s=\hat{\vec{n}}-\vec{\beta}$, and

$$\frac{\partial s}{\partial t_{\text{ret}}}=\frac{\partial R}{\partial t_{\text{ret}}}-\frac{1}{c}\left(\frac{\partial\left(R\hat{\vec{n}}\right)}{\partial t_{\text{ret}}}\cdot\vec{v}+R\hat{\vec{n}}\cdot\frac{\partial\vec{v}}{\partial t_{\text{ret}}}\right)=-\hat{\vec{n}}\cdot\vec{v}-\frac{1}{c}\left(-v^2+R\hat{\vec{n}}\cdot\dot{\vec{v}}\right)=-\hat{\vec{n}}\cdot\vec{v}-\frac{1}{c}\left(-v^2+R\hat{\vec{n}}\cdot\dot{\vec{v}}\right)$$

Substituting these values in the equation for $\vec{E}$, we write $\vec{E}$ as a sum of two terms, the first being independent of acceleration, and hence necessarily identical to the field we previously derived for a charge moving at constant velocity:

For constant velocity,

$$\vec{E} = \frac{q}{4\pi\varepsilon_0}\left(\vec{R} - \frac{R\vec{v}}{c}\right)\frac{1}{s^3}\left(1 - \frac{v^2}{c^2}\right)$$

with $s = R\left(1 - \vec{\beta}\cdot\hat{\vec{n}}\right)$.

With acceleration, there is an additional term we call $\vec{E}_{\text{rad}}$:

$$\begin{aligned}\vec{E}_{\text{rad}} &= \frac{q}{4\pi\varepsilon_0}\left[\frac{\vec{R}c - R\vec{v}}{c^2 s^3}\frac{1}{c}\vec{R}\cdot\dot{\vec{v}} - \frac{R}{s^2c^2}\dot{\vec{v}}\right] \\ &= \frac{q}{4\pi\varepsilon_0}\frac{1}{c^2 s^3}\left[\left(\vec{R} - \frac{R\vec{v}}{c}\right)\vec{R}\cdot\dot{\vec{v}} - Rs\dot{\vec{v}}\right]\end{aligned}$$

It is easy to check that this can be written:

$$\vec{E}_{\text{rad}} = \frac{q}{4\pi\varepsilon_0}\frac{1}{c^2 s^3}\left[\vec{R}\times\left[\left(\vec{R} - \frac{R\vec{v}}{c}\right)\times\dot{\vec{v}}\right]\right].$$

The reason this term is called $\vec{E}_{\text{rad}}$ is that it decreases with distance as $1/R$, in contrast to the field from a charge moving at constant velocity, so $\vec{E}_{\text{rad}}$ carries a finite energy away from the particle—it is a radiation field. The accompanying magnetic field can be found similarly, and is $\vec{B}_{\text{rad}} = \hat{\vec{n}}\times\vec{E}_{\text{rad}}/c$.

The final expression for the electric field, combining the two terms above, and restoring $s = R\left(1 - \vec{\beta}\cdot\hat{\vec{n}}\right)$ is:

$$\vec{E} = \frac{q}{4\pi\varepsilon_0 R^2}\left[\frac{\hat{\vec{n}} - \vec{\beta}}{\gamma^2\left(1 - \vec{\beta}\cdot\hat{\vec{n}}\right)^3}\right]_{\text{ret}} + \frac{q}{4\pi\varepsilon_0 c}\left[\frac{\hat{\vec{n}}\times\left[\left(\hat{\vec{n}} - \vec{\beta}\right)\times\dot{\vec{\beta}}\right]}{\left(1 - \vec{\beta}\cdot\hat{\vec{n}}\right)^3 R}\right]_{\text{ret}}.$$

4/18/18

20. Thomas Precession

Introduction: Spin-Orbit Coupling and Precession

If an atom is placed in a magnetic field, the spectral lines split up into series of close-together lines. The first explanation, before quantum mechanics, was offered by Larmor and Zeeman: imagine an electron in a circular orbit around a proton. Now add a uniform magnetic field. If the orbit is perpendicular to the field, this adds another central force, raising or lowering the frequency depending on the direction of the circular motion. If the field lines are parallel to the orbit, the frequency change would be negligible. This would suggest why there might be three lines, as were indeed observed in some atoms. However, for other atoms more lines were observed, for example five. This was termed the *anomalous Zeeman effect*, and was easily explained a little later with the advent of quantum mechanics: it arose from the quantization of the component of angular momentum along the field direction.

(*Aside*: The Zeeman effect is used in astronomy to measure magnetic fields in stars, for example in a sunspot, and is also important in fine tuning laser Doppler slowing of atoms to catch them in a magneto optical trap at very low temperatures.)

But that wasn't the end of the story. More line splittings were observed which could only be explained by assuming the electron itself had an intrinsic angular momentum, called spin, of value $s_z = \pm\frac{1}{2}\hbar$, and accompanying magnetic moment $\vec{\mu} = \left(ge/2mc\right)\vec{s}, \quad g = 2.$ This gave the line splittings from the magnetic field correctly, but did not properly account for the spin-orbit interaction, the energy of the electron's intrinsic magnetic moment in the field created by its orbital motion, a little circle of current.

To understand the spin-orbit interaction, we'll begin with the trivial case of an electron at the origin in a magnetic field $\vec{B} = b\hat{\vec{z}}.$ It has a magnetic energy $U = -\vec{\mu}\cdot\vec{B},$ and will precess, $d\vec{s}/dt = \vec{\mu}\times\vec{B}$, a frequency $\omega = geB/2mc.$

To apply this analysis to the electron circling around the proton (we'll just do hydrogen for now), if at some instant the electron has velocity $\vec{v}$, then to leading order in $\left(v/c\right)$ the electron sees a magnetic field in its own rest frame of $\vec{B} - \left(\vec{v}/c^2\right)\times\vec{E},$ where $e\vec{E} = -\left(\partial V/\partial r\right)\vec{r}/r,$ $V\left(r\right)$ being the potential energy of the electron in the proton's field.

Thus the effective orbital magnetic field is given as

$$-\left(\vec{v}/c^2\right)\times\vec{E} = \left(\vec{v}/c^2\right)\times\frac{\vec{r}}{er}\frac{\partial V}{\partial r} = -\left(\vec{r}\times m\vec{v}\right)\frac{1}{mec^2}\frac{1}{r}\frac{\partial V}{\partial r}.$$

This gives an expression for the electron's total spin energy, and therefore for its precessional frequency,

$$\begin{aligned} U' &= -\vec{\mu}\cdot\left(\vec{B} - \left(\vec{v}/c^2\right)\times\vec{E}\right) \\ &= -\frac{ge}{2mc}\vec{s}\cdot\vec{B} + \frac{g}{2m^2c^3}\left(\vec{s}\cdot\vec{L}\right)\frac{1}{r}\frac{dV}{dr}. \end{aligned}$$

The problem is that this is *wrong*: the $\vec{s}\cdot\vec{L}$ spin-orbit term (which, notice, is a relativistic effect) is too large by a factor of 2 compared with that observed experimentally!

(Here $g = 2$ and the spin precessional frequency about the angular momentum is given by the coefficient).

So, what's wrong with the argument that led to this?

The flaw is that to find the field in the electron's rest frame, we had to do a boost from the lab frame. The spin orbit term so generated will give a precession frequency, but the problem is that the electron is constantly accelerating in a direction perpendicular to its motion, so to find the precession frequency we need to take two successive boosts differing incrementally—but a product of two boosts is not a pure boost, it includes some rotation, so that must be added into the rotational frequency from the energy found above, and in fact cuts the orbital contribution by a factor of two. This result, found by Thomas in 1926, matches the experimental findings.

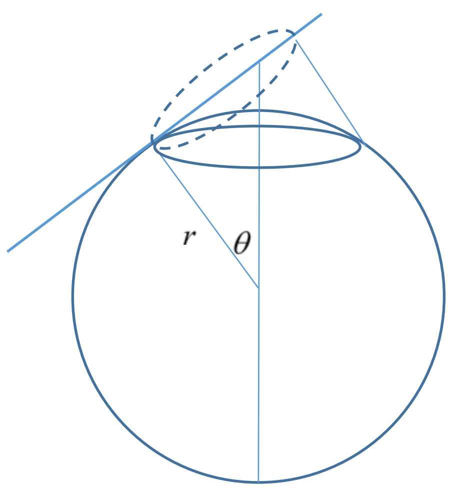
Dealing with a Succession of Relatively Rotated Frames

(*Mostly following M. I. Krivoruchenko*, arXiv:0805.1136v2 [nucl-th])

As the electron circles the proton, the succession of frames in which it is momentarily at rest circle with its momentum vector. *If no magnetic field is present, the spin will as far as possible stay pointing in the same direction.*

This is analogous to the famous problem of parallel transporting a vector on a spherical surface: even if the incremental transport keeps it parallel, on going around a finite closed path the arrow won't end up pointing in its original direction. This is most easily seen by considering going around an octant, from the north pole down to the equator, with an arrow pointing along the path, then a quarter of the way around the equator, the arrow pointing now due south, then straight back up to the north pole. On getting back, the arrow will be $\pi/2$ off from its original direction, despite being parallel transported at each step.

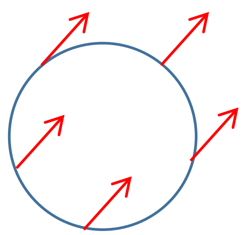
A classic example, amazingly similar to Thomas precession, is Foucault's pendulum, so we'll review that first.



The dashed ellipse is the projection into the tangent plane of the circle of constant latitude.

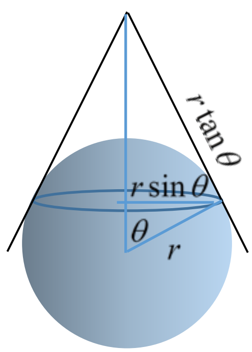
Foucault's Pendulum

Foucault's pendulum famously demonstrated the rotation of the Earth: it swung in a fixed plane, relative to the stars, as the Earth rotated beneath it. At colatitude θ (meaning we take the physicist's notation, $\theta = 0$ at the north pole) the component of the earth's angular momentum perpendicular to the ground is $\omega\cos\theta$, and this is the observed rate of rotation of the pendulum's plane of swing.



Parallel transport around a circle in a plane

We'll now analyze this from a slightly different point of view: in 24 hours, we see the pendulum as being transported (by the earth's rotation) along a path on the spherical surface going around one circle of latitude. As it is moved along, we require "parallel transport", meaning that we constantly adjust it to keep it pointing in the same direction. The question is, to what extent does it turn around anyway? We've seen that on a spherical surface, it does turn, and in fact it can be proved that it turns through an angle equal to the solid angle enclosed by the path.
But on a flat surface, we can easily adjust the direction at each stage to keep it pointing the same way, so at the end it is still the same as at the beginning.
Notice from the diagram that on taking the arrow around a circle in a plane, the arrow always points the same way, but if we think of the angle between the circle and the path, that turns through 2π. We have to be clear which angle we're referring to.
On the surface of a sphere, parallel transport happens on moving along a segment of a great circle (meaning centered at the center of the sphere). We could imagine going around at a constant latitude—*not* a great circle—as made up of many segments of great circles, but they would intersect at nonzero angles, where we'd have to adjust appropriately.



Cone hat on sphere, touching along latitude circle

To see how much our latitudinal trip turns the arrow, let's take the sphere to have radius r, then the trip has length $2\pi r\sin\theta$. Now think about a tiny part $(r\sin\theta)\Delta\phi$ of the path. It's a very good approximation to take this segment to be in the tangent plane to the sphere at the point in the middle of the segment. The segment of the latitude circle projects into this plane as the end of an ellipse: the ellipse has one axis the same as the latitude circle ($r\sin\theta$), the other shortened by a factor $\cos\theta$ (see figure).
The radius of curvature of the latitude circle is $r\sin\theta$, the local radius of curvature r_C of its projection into the flat tangent plane is (larger) $r\sin\theta/\cos\theta = r\tan\theta$. This means that as we move an arrow a small distance ds along this path (which for increments is the same as the path in the local flat tangent plane) t*o keep it pointing in a constant direction* we must rotate it through an angle $ds/r_C = (ds/r)\cot\theta$ relative to the path. Therefore, the angle turned through in going all the way around the sphere is $2\pi r\sin\theta(1/r)\cot\theta = 2\pi\cos\theta$, again relative to the path.
Now for a circle of latitude, on going completely around, the direction of the path relative to the fixed stars goes through 2π. So on approaching a small circle around the north pole, comparing the pendulum's direction with *a fixed line in the rotating plane*, the angle turned through per day approaches 2π (it's actually $2\pi\cos\theta$), but relative to the fixed stars, in 24 hours the Foucault pendulum will have turned through an angle of $2\pi(1-\cos\theta)$, going to zero at the pole.

The Cone Hat Trick

An easier way to derive this result (which we'll also use for Thomas precession) is to place a conical hat on the sphere, just touching the sphere along the line of latitude we are going to travel around. Where they touch, the sphere and cone have a common tangent plane, so incrementally moving the arrow along the latitude, keeping its direction constant, is the same as doing it on the cone.

But, unlike the sphere, the cone can be unfurled and laid out flat. This makes it easy to see what parallel transport means: just keep the arrow pointing in the same direction in the plane. Relative to the line of latitude, then, such an arrow will have changed direction by ϕ on going once around, where

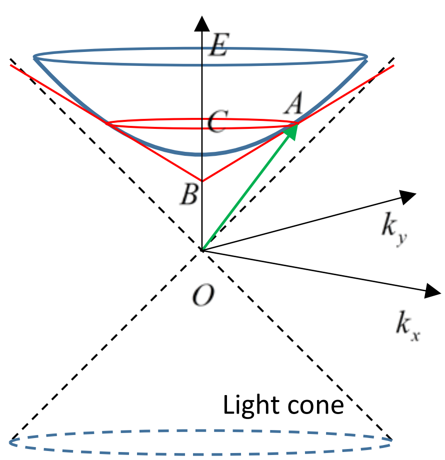
Flattened cone: radius is $r\tan\theta$, circumference $2\pi r\sin\theta$.

$$\phi = 2\pi - \frac{2\pi r\sin\theta}{r\tan\theta} = 2\pi(1-\cos\theta).$$

This agrees with our earlier finding.

Thomas Precession

The analysis of Thomas precession is remarkably similar to that above, and the cone trick works again. We're trying to track the spin direction as the electron orbits around the proton. Remember, we are not including the precession from the magnetic field, that can be added later. In our model, there is *no physical torque* on the spin, so it will be parallel transported from one frame to the next as the electron moves around. In fact it will precess, as we've said, but it is a purely kinematical effect, arising from the successive boosts necessitated by the acceleration, the product of boosts giving rotation. Our job is to calculate how much spin rotation comes about as a result of one orbit period.



As the electron circles around the proton, it is also moving around a circle in velocity space, conveniently parametrized with rapidity ψ. Suppose at $t=0$ its 4-velocity is $\overrightarrow{OA} = \left(\gamma, \hat{\vec{n}}\gamma\beta\right)$. The hyperboloid (looking like a saucer) surface is the set of 4-velocities (for the three-velocity confined to the (x, y) plane).

Now we imitate the analysis we just did for Foucault's pendulum. The red circle, and red cone touching the hyperboloid along that circle, is equivalent to the cone hat we used above. We label the apex of the touching cone B. Finally, we label the center of the red (touching) circle C. At this point, one might think we just have to unroll and flatten out the cone hat, just like the Foucault case, and we're done. In fact, this is correct, but now it's a bit more tricky because the diagram above is in Minkowski space, not a Euclidean metric, so extra care is needed. We see that since $\overrightarrow{OA} = \left(\gamma, \hat{\vec{n}}\gamma\beta\right)$, then

$\overrightarrow{CA}=\left(0,\hat{\vec{n}}\gamma\beta\right)$. Clearly, $\overrightarrow{BA}$ has the same spatial momentum component, but how do we find its energy component?

The key is that $\overrightarrow{BA}$ is a tangent to the hyperboloid, and therefore perpendicular to any line from the center to the point of tangency (just like for a sphere). That is,

$$\overrightarrow{BA}\cdot\overrightarrow{OA}=0,$$

Remembering that this is a Minkowskian inner product! If those vectors don't look orthogonal, that's because you're thinking Euclidean. Think of the x',t' axes after a Lorentz transformation. They're still orthogonal. And, a vector on the light cone is orthogonal to itself.

From the above equation,

$$\overrightarrow{BA}=\left(\gamma\beta^2,\hat{\vec{n}}\gamma\beta\right).$$

It follows that these space-like vectors have lengths

$$\left|\overrightarrow{BA}\right|=\beta=\tanh\psi,\quad \left|\overrightarrow{CA}\right|=\beta\gamma=\sinh\psi.$$

Flattened cone: radius is $\tanh\psi$, circumference $2\pi\sinh\psi$.

By exact analogy with the Foucault pendulum, we find the precession in one revolution is

$$\phi=2\pi\left(1-\cosh\psi\right).$$

In the low energy limit (relevant for atoms) the Thomas angular velocity

$$\vec{\omega}_T=\left(\frac{1-\gamma}{\beta^2}\right)\dot{\vec{\beta}}\times\vec{\beta}$$

becomes

$$\omega_T=\tfrac{1}{2}\frac{\vec{a}\times\vec{v}}{c^2}.$$

The acceleration is from the screened Coulomb field,

$$m\vec{a}=-e\frac{\vec{r}}{r}\frac{dV}{dr}$$

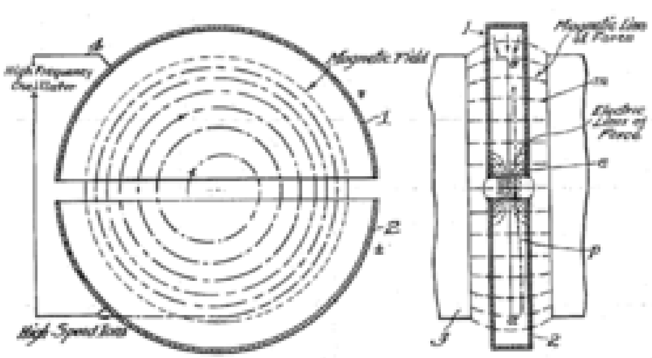
so this is equivalent to an additional magnetic field term, and

$$\vec{\omega}_T = -\frac{1}{2c^2}\frac{\vec{r}\times\vec{v}}{m}\frac{1}{r}\frac{dV}{dr} = -\frac{1}{2m^2c^2}\vec{L}\frac{1}{r}\frac{dV}{dr}.$$

Comparing with the spin orbit term found near the beginning of the lecture, since $g = 2$ this subtracts half the term, to give agreement with experiment.

21. Synchrotron Radiation

Cyclotrons and Synchrotrons



Apart from high voltage electrostatic machines, the first machines to accelerate electrons and protons beyond kilovolt energies were *cyclotrons* (patented by Lawrence in 1934, this is his patent application, in Wikipedia) in which the particle moves in a planar spiral in a perpendicular magnetic field, given an electric kick every half turn, the increase in speed increases orbit size, but (nonrelativistically) the orbit *time* is constant, so an AC field can be used to provide the acceleration. The particles begin near the center of the machine, and come out at the maximum radius. However, to increase speed into the relativistic range, the increase in mass means that the orbital time is no longer independent of energy. Some lagging can be tolerated, but obviously not much.

Therefore, the particles must be fed into the machine in bunches (rather than continuously) and the timing of electrical impulses then synchronized with the orbital time increasing with energy. This is the *synchrotron*. The limiting factor for increasing energy in a synchrotron is that the particles circling are of course accelerating towards the center and therefore radiating energy.

Perhaps the most important synchrotron radiation facility is the Advanced Photon Source (APS) at Argonne National Lab. There electrons are accelerated to 7 Gev ($\gamma \simeq 14,000$) and then sent into storage rings (1 km diameter) where rf fields are used to maintain their energy. X-rays from this machine have recently made major contributions to finding the structure of important proteins, for example.

Summarizing Radiation Formulas

To analyze synchrotron radiation, we'll first summarize here the radiation formulas found so far.

First, the radiated power:

The nonrelativistic Larmor formula:

$$P = \frac{1}{6\pi\varepsilon_0} \cdot \frac{a^2 q^2}{c^3} = \frac{\mu_0}{6\pi} \cdot \frac{a^2 q^2}{c},$$

and the relativistic generalization:

$$P = \frac{\mu_0 c q^2 \gamma^6}{6\pi} \left[\dot{\beta}^2 - \left(\beta \times \dot{\beta} \right)^2 \right]_{\text{ret}}.$$

In linear motion, the second term of course disappears.
In circular motion at radius r:

$$P = \frac{\mu_0}{6\pi} \cdot \frac{q^2 c^3}{r^2} \beta^4 \gamma^4.$$

To find the pattern of radiation needs significantly more work. We need the Lienard-Weichert formula for the electric field, the magnetic field radiated has essentially the same form, so we can find the Poynting vector in any direction.
The electric field is:

$$\vec{E} = \frac{q}{4\pi\varepsilon_0 R^2} \left[\frac{\hat{\vec{n}} - \vec{\beta}}{\gamma^2 \left(1 - \vec{\beta} \cdot \hat{\vec{n}} \right)^3} \right]_{\text{ret}} + \frac{q}{4\pi\varepsilon_0 c R} \left[\frac{\hat{\vec{n}} \times \left[\left(\hat{\vec{n}} - \vec{\beta} \right) \times \dot{\vec{\beta}} \right]}{\left(1 - \vec{\beta} \cdot \hat{\vec{n}} \right)^3} \right]_{\text{ret}}.$$

The first term is just the relativistic static field from the moving charge, the same as for a nonaccelerating (but moving) charge. The second term is the radiation.

Circular Acceleration: Angular Distribution

Following Jackson, we take $\vec{\beta} = \beta \hat{z}, \quad \dot{\vec{\beta}} = \dot{\beta} \hat{x}, \quad \hat{n} = \sin\theta\cos\phi\,\hat{x} + \sin\theta\sin\phi\,\hat{y} + \cos\theta\,\hat{z}$ in

$$\frac{dP(t_{\text{ret}})}{d\Omega} = \frac{\mu_0 c q^2}{16\pi^2} \frac{\left| \hat{\vec{n}} \times \left[\left(\hat{\vec{n}} - \vec{\beta} \right) \times \dot{\vec{\beta}} \right] \right|^2}{\left(1 - \vec{\beta} \cdot \hat{\vec{n}} \right)^5} \Bigg|_{\text{ret}}.$$

to find with lots of algebra

$$\frac{dP(t_{\text{ret}})}{d\Omega} = \frac{\mu_0 q^2 \dot{v}^2}{16\pi^2 c} \cdot \frac{1}{\left(1 - \beta\cos\theta \right)^3} \cdot \left[1 - \frac{\sin^2\theta\cos^2\phi}{\gamma^2 \left(1 - \beta\cos\theta \right)^2} \right] \Bigg|_{\text{ret}}.$$

Recall now β is very close to one, $1 - \beta \sim 10^{-8}$, so this beam is well focused in the θ direction. Huge power per unit solid angle, but over a very small solid angle.

The Heaviside-Feynman Formula

The expression for the electric field of a moving point charge given above can be written in a quite different way (the derivation is nontrivial) first found by Heaviside, discovered independently by Feynman:

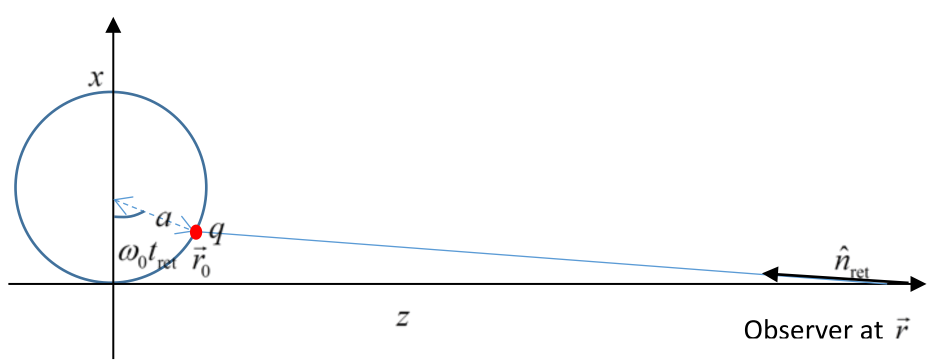
$$\vec{E}(\vec{r},t)=\frac{q}{4\pi\varepsilon_0}\left\{\left[\frac{\hat{n}}{R^2}\right]_{\text{ret}}+\frac{R_{\text{ret}}}{c}\frac{d}{dt}\left[\frac{\hat{n}}{R^2}\right]_{\text{ret}}+\frac{1}{c^2}\frac{d^2\hat{n}_{\text{ret}}}{dt^2}\right\},$$

and the magnetic field is given by

$$c\vec{B}(\vec{r},t)=\hat{n}_{\text{ret}}\times\vec{E}(\vec{r},t).$$

(The formula is given without derivation in Volume 1 of Feynman's Lectures, 28-2. It's derived in Zangwill, and given, naturally, as a very long problem in Jackson.)
The first term is the ordinary Coulomb field from the retarded time. The second term more or less updates the field to the one we are now familiar with, the "squashed" Coulomb field for a charge moving at constant velocity, centered at the actual position of the charge. The third term includes the radiative field from the charge's acceleration, notice it's the acceleration *in the observer's time* of the unit vector pointing to the retarded particle position. But there's more: it is a unit vector, so it's accelerating even if moving uniformly round a circle, and it will in general be accelerating along the circle for a charge in steady linear motion. Since the whole formula is exact, these contributions must be cancelled by the second term, to ensure no radiation from a steadily moving charge.
In fact, though, these contributions are negligible if the accelerating charge is in a localized region smaller than the distance to the observer. Note also that since the third term is a unit vector, the $1/R$ amplitude dependence is automatically taken care of.

Synchrotron Radiation



The circle represents the synchrotron, radius a, and a charge q at angle $\omega_0 t_{\text{ret}}$. The observer is taken to be very far away in the z direction. The *unit* vector $\hat{n}_{\text{ret}}$ points from the observer to the apparent position of the charge, that is, where it was at the retarded time—in other words, where the observer *sees* it to be.

$$\vec{r}_0(t_{\text{ret}}) = a(1-\cos\omega_0 t_{\text{ret}})\hat{x} + a\sin(\omega_0 t_{\text{ret}})\hat{z}.$$

Now the vector $\hat{n}_{\text{ret}} = [\vec{r} - \vec{r}_0(t_{\text{ret}})]/R_{\text{ret}}$ where $\vec{R} = \vec{r} - \vec{r}_0,$ and to a good approximation $R_{\text{ret}} = r - a\sin(\omega_0 t_{\text{ret}}),$ so

$$\hat{n}_{\text{ret}} = \frac{a(\cos\omega_0 t_{\text{ret}} - 1)\hat{x} + [r - a\sin(\omega_0 t_{\text{ret}})]\hat{z}}{r - a\sin(\omega_0 t_{\text{ret}})}.$$

Notice now the $\hat{z}$ component is time-independent, and in the $\hat{x}$ component we can just keep the r in the denominator, so Feynman's formula for the radiating electric field becomes

$$\vec{E}(\vec{r},t) = \frac{q}{4\pi\varepsilon_0 c^2}\frac{d^2\hat{n}_{\text{ret}}}{dt^2} = \frac{q\hat{x}}{4\pi\varepsilon_0 c^2}\frac{d^2}{dt^2}\frac{a\cos\omega_0 t_{\text{ret}}}{r}.$$

Now

$$c(t - t_{\text{ret}}) = r - a\sin(\omega_0 t_{\text{ret}}),$$

Rearranging, and using $\beta c = a\omega_0,$ and incorporating the constant r/c into the observer's clock setting,

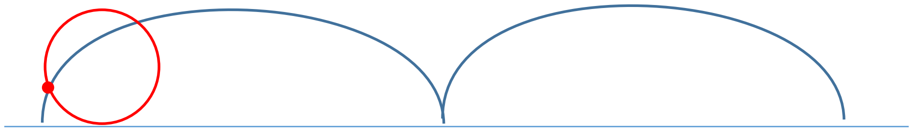
$$\omega_0 t = \omega_0 t_{\text{ret}} - \beta\sin(\omega_0 t_{\text{ret}}).$$

Again use $\beta c = a\omega_0$ to write

$$\vec{E}(\vec{r},t) = \frac{q\hat{x}}{4\pi\varepsilon_0 c\omega_0}\frac{d^2}{dt^2}\frac{\beta\cos\omega_0 t_{\text{ret}}}{r}.$$

These two equations describe a curve parameterized by t_{ret}. We need to plot $\beta\cos\omega_0 t_{\text{ret}}$ against $\omega_0 t$ and find its second derivative to find $\vec{E}(\vec{r},t)$ as a function of observer time t.
In fact, this is a generalized cycloid: imagine rolling a wheel of unit radius along a horizontal line, and track a point on the wheel a distance β from the center. Now t_{ret} is how long the wheel has been rolling, with angular velocity ω_0. Its center will have moved through $\omega_0 t_{\text{ret}}$ horizontally, the point is horizontally $-\beta\sin(\omega_0 t_{\text{ret}})$ from the center, and vertically displaced by $\beta\cos(\omega_0 t_{\text{ret}})$.
If the spot is at the center of the wheel, $\beta = 0,$ the radiated field is zero. If $\beta \ll 1,$ the electric field oscillates sinusoidally with frequency ω_0. But as $\beta \to 1,$ the curve approaches an actual cycloid,

tending to a cusp, the second derivative goes to infinity. The radiation becomes a series of pulses.



For a better cycloid, click here!

Note on this cycloid curve: the standard notation would be $x=\theta-\beta\sin\theta,\ y=1-\beta\cos\theta,$ with almost cusp at $\theta=0$ for $\beta\to 1.$ Near the origin, $x\cong(1-\beta)\theta,\ y\cong 1-\beta+\frac{1}{2}\theta^2$ so

$d^2y/dx^2\cong 1/(1-\beta)^2.$

This tells us the curvature at the cusp compared with the curvature of the rest of the curve (of order unity) is up by a factor of γ^4.

The Advanced Photon Source has radius one kilometer, so the circling frequency is of order 100 kHz. At that frequency, the wavelength radiated would be of order a kilometer. But near the cusp this is shortened by a factor of order γ^4, and $\gamma\sim 10^4$, and the radiation is X-rays.

Cosmic Synchrotron Radiation

Synchrotron radiation is common in astronomy. Of course, it signals the presence of magnetic fields, so gives extra insight into the systems being studied. It has helped understand Jupiter's magnetosphere. It is present in the remnants of supernovae, electrons in a magnetic field give fairly well polarized bright radiation, but also are transparent. Synchrotron radiation from neutron stars suggests magnetic fields of order 10^8 tesla.

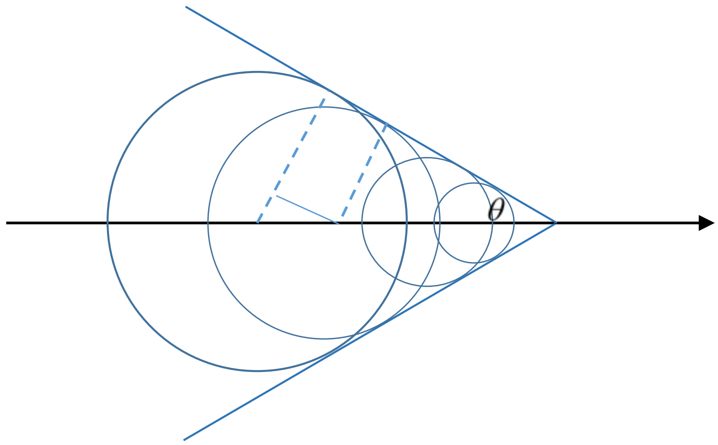
22. Cherenkov Radiation

Introduction

When a charged particle moves through a transparent medium at a speed greater than the phase velocity of light in the medium, radiation is emitted. This is called Cherenkov radiation, after Pavel Cherenkov, who was the first to detect it experimentally in 1934. He received the Nobel Prize for this discovery, shared with the two theorists, Tamm and Frank, who explained the effect. (In fact, it had been predicted by Heaviside in 1888.)

A simple picture of Cherenkov radiation is provided by comparing it with the shock wave generated by a supersonic airplane. (Check out this animation.) The sound waves emitted at each instant radiate outwards in a spherical wave in the air, the plane outstrips the outgoing wave and continues to generate spheres, the envelope of these spheres is the supersonic shock wave, a cone, with semi angle given by

$$\sin\theta = \frac{v_{\text{sound}}}{v_{\text{plane}}}.$$



It's called the Mach cone.

It's interesting to note that the waves along the shock front are *in phase*: look at the path length difference between the two dashed lines, and the time variation of the emissions, a d the ratio of velocities.

Going now to the analogous problem of a charged particle moving through a dielectric material faster than the speed of light in that material, the first point to make is that the radiation loss is small, so to a very good approximation the particle is moving at constant velocity, so the particle itself is not directly responsible for the radiation, only indirectly, in that the moving charge causes a localized dielectric shielding response, obviously time-dependent, and this is the excitation that travels outwards from the immediate vicinity of the particle.

We can analyze this in the usual way, the potential and field observed at current time t at displacement $\vec{R}$ from the current particle position comes from the particle + shielding at t_{ret}, the observer being at $\vec{R}_{\text{ret}}$ relative to the particle position at that time (see drawing) so

$$t - t_{\text{ret}} = \frac{R_{\text{ret}}}{c/n}.$$

However, this is different from our earlier analysis of fields from a moving charge. Glancing at the figure above, we can see that there are points in the cone where two of the circles intersect, such a point will see two signals at the same instant from the moving charge at different earlier times! In fact, this is true for *any* point *in* the cone: there is really a continuous family of these circles, beginning with tiny ones near the point of the cone, think of the continuously expanding circles, given a point inside the cone, eventually a circle will pass through it, then it will be inside subsequent circles until it is on the boundary of one.

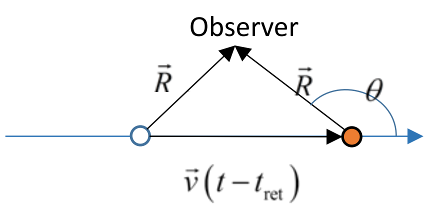
Or, in line with our previous analysis of the world line of a particle always being inside the light cone, the "light cones" in the dielectric are narrower, and a particle moving faster than the speed of light in the medium can intersect the cone twice (at a steady velocity).

The two points of intersection coincide for an observation point *on* the cone, so there is a heavy concentration of energy there.

To find the electric and magnetic fields, the Lienard-Weichert potentials still work if we replace $\varepsilon_0 \to \varepsilon,\ \mu_0 \to \mu,\ \vec{\beta}_n = \vec{v}/c_n = \vec{v}/(c/n)$. We'll assume for now that ε, μ are frequency-independent, so we can write

$$\varphi(\vec{r},t) = \frac{1}{4\pi\varepsilon_0}\left[\frac{q}{R-\vec{\beta}_n\cdot\vec{R}}\right]_{\text{ret}}, \quad \vec{A}(\vec{r},t) = \left[\frac{q\vec{v}}{R-\vec{\beta}_n\cdot\vec{R}}\right]_{\text{ret}}.$$

Notice we have to put the observer inside the Mach cone, or nothing will be observed.



From the diagram, and squaring $R_{\text{ret}} = \left|\vec{R} + \vec{v}(t-t_{\text{ret}})\right|$, we find a quadratic equation for $t-t_{\text{ret}}$, with roots

$$t - t_{\text{ret}} = \frac{-\vec{v}\cdot\vec{R} \pm \sqrt{(\vec{v}\cdot\vec{R})^2 - (v^2-c_n^2)R^2}}{v^2-c_n^2} \geq 0.$$

Now for Cherenkov radiation, we also need $\vec{v}\cdot\vec{R} < 0, \quad \sin\theta \leq \sin\theta_C = 1/\beta_n$. (This last being when the two roots coincide).

Zangwill finds $\varphi, \vec{A}$ by feeding in these two values, and so can find the fields. He finds the electric field indeed is singular on the shockwave front, but its nonzero value inside the cone points the wrong way—towards rather than away from the charge. There is a simple explanation for this: as the charge passes, it polarizes the medium, but it's moving so fast that immediately behind it is a net negative charge density that takes time to relax, in fact it is part of the outwardly propagating wave. However, Wikipedia currently doesn't like this picture, I'm not sure why.

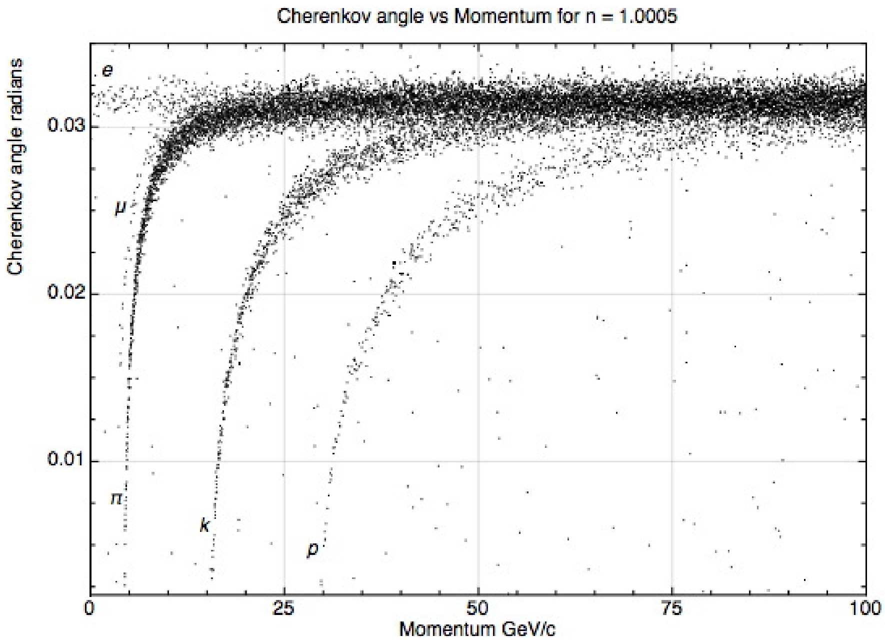
The sharp nature of the shock wave front (which is actually somewhat fuzzed out by frequency variations in dielectric response, which we've ignored) suggests that the radiation covers a wide range of frequencies, and over the visible range it tends if anything to increase with frequency, so the light appears blue.

Uses of Cherenkov Radiation

Widely used in biochemistry and medicine, radioactive markers emit energetic particles which are traveling faster than light in biomaterials (essentially, water). Cherenkov radiation is used to detect them.

Used in neutrino searches: example, IceCube, at the south pole. One cubic kilometer of ice, with many Cherenkov radiation detectors. Finds a few dozen neutrinos a year.

Used in many high energy experiments to find velocity of particles of known momentum, hence find their mass, in identifying decay fragments from collisions. The dielectric medium is typically a gas, with $\varepsilon = 1.0005\varepsilon_0$, say, so there won't be any Cherenkov radiation if the particle is traveling slower than $c/1.0005, \quad \gamma < 30$ or so.



Various Generators of Radiation

Particles accelerated in a synchrotron can be caused to radiate by going linearly through a *wiggler*. This is a sequence of alternating magnets causing the path to deviate from a straight line to a sinusoidal path. (In practice, this is a straight section in the storage "ring", clearly not a circular ring.) The consequent back and forth oscillation generates dipole radiation, the frequency determined by particle speed and magnet spacing.

The *free electron laser* (FEL) is essentially a wiggler with reflecting mirrors at the two ends, so the radiation intensity builds coherently to a strength where there is ponderomotive feedback on the particle current.

Transition radiation (TR) is generated when a fast particle goes into a material with a different refractive index. The electric field lines will be changed, and connecting the old lines to the new ones will generate radiation, analogous to Purcell's analysis of radiation from a nonrelativistic accelerating charge.

The *Smith-Purcell Effect* is the generation of radiation by a fast charged particle moving just above a metal diffraction grating. The sequential response of the strips of metal will generate outgoing coherent radiation at certain angles. The is currently used for terahertz radiation, a frequency range difficult to generate otherwise.

23. Bremsstrahlung

Just a few remarks…

We've analyzed the radiation emitted by an accelerating particle, and how that limits accelerator design. A closely related topic, of arguably equal importance, is the radiation emitted by a *decelerating* particle. The formulas are the same ones we have found, the radiation is called bremsstrahlung, the German for braking radiation, since it is emitted as the charged particle decelerated on interacting with other particles.

The formulas for this radiation are the same ones we've been using—recall, for example, that non-relativistically the radiation goes as the *square* of the acceleration.
The first use of bremsstrahlung was X-rays, accidently discovered in the 1890's when a high voltage (some kilovolts) discharge tube (like a neon sign) was found to spoil boxed photograph film kept nearby. The rays came from energetic electrons hitting the metal anode and rapidly decelerating on being deflected by the electric fields of the nuclei. The relativistic correction is rather small at these energies, so the Larmor formula is reasonably good. (Typical X-ray spectra also include some sharp lines, from inner shell electrons being ejected, then replaced with other electrons making the transition down.)
One way to see what's going on is to imagine the electron as one end of a dipole, anchored at the nucleus.
What about the fast electrons hitting slow electrons? If it's head on, there is little change in the fields, if not, think about the center of mass: the two dipoles cancel, leaving much smaller quadrupole radiation.
For very fast charged particles suddenly stopped, we can use Purcell's picture to get an idea: at some instant Δt shortly after the crash, outside a sphere of radius $c\Delta t$ the pancake Coulomb field is moving forward at almost the speed of light. Connecting it to the spherically symmetric Coulomb field of the stopped particle takes a lot of electrical field energy, radiating outwards.
Then, beta decay: a fast particle (an electron) shoots from a nucleus which is suddenly a positive charge. It's like tearing part a suddenly created dipole. Obviously, there is radiant energy to be included in the overall energy bslance.

Plasma cooling, including opacity

A hot plasma cools down mainly by bremsstrahlung emission. This is sometimes called "free, free" emission, since both charged particles are free to move, two electrons scattering each other will both emit. This is in contrast with emission in an X ray tube, where the much heavier nuclei, responsible for most of the electron deflection, scarcely move themselves, so emit negligible amounts of radiation.
In clusters of galaxies, maybe 15% of the total mass is in hot gas away from stars. Bremsstrahlung is observed from the region around and between the galaxies. These regions are transparent to galaxies further away, so cannot be hot dust clouds.
In a tokomak type plasma, bremsstrahlung would cause rapid cooling, but the plasma is far from transparent, so much of this energy is reabsorbed.

24. Relativistic Dynamics III

Principle of Least Action

(*Following Landau—this first section is a reminder of earlier work*)
For any mechanical system there is an integral S, called the action, which has a minimum value for the path traced in configuration space as the system evolves in time, as opposed to other paths between the initial and final configurations.
The action integral for a free material particle cannot depend on the coordinate system chosen, it must be Lorentz invariant. The only possibility for a free particle with no external force is integrating over the infinitesimal intervals

$$S=-\alpha\int_a^b c\,d\tau,$$

with τ the proper time, and α some yet to be determined constant. Since we know a moving clock runs slow, we see that in the inertial frame in which the beginning event a and the final event b are both at the spatial origin, the path corresponding to staying at that origin is the extremum. Actually it's the *longest* possible time, so we put in the minus sign.

The standard notation for the action in mechanics is an integral over time,

$$S=\int_{t_1}^{t_2} L\,dt,$$

where L is the Lagrangian. Since we know that $d\tau=\gamma dt,$ evidently

$$L=-\alpha c\sqrt{1-\frac{v^2}{c^2}}\approx-\alpha c+\frac{\alpha v^2}{2c}.$$

We know that non-relativistically, $L=\frac{1}{2}mv^2,$ and the first constant $-\alpha c$ is irrelevant in minimizing, so we have

$$S=-mc^2\int_a^b d\tau,$$

and

$$L=-mc^2\sqrt{1-\frac{v^2}{c^2}}.$$

The momentum

$$\vec{p}=\frac{\partial L}{\partial\vec{v}}=\frac{m\vec{v}}{\sqrt{1-v^2/c^2}}.$$

The energy is defined as

$$E=\vec{p}\cdot\vec{v}-L=\frac{mc^2}{\sqrt{1-v^2/c^2}}.$$

The principle of least action is that along the physical path

$$\delta S=-mc\int_a^b d\tau=0.$$

Writing $d\tau = \sqrt{-dx_i dx^i}$ (always real, a particle must follow a timelike path) we have

$$\delta S = mc\int_a^b \frac{dx_i \delta dx^i}{d\tau} = mc\int_a^b u_i d\delta x^i ,$$

and integrating by parts

$$\delta S = -mcu_i \delta x^i \Big|_a^b - mc\int_a^b \delta x^i \frac{du^i}{d\tau} d\tau.$$

Taking arbitrary δx^i, except zero at the ends, we find the four-acceleration $du^i / d\tau = 0$ along the path, as we expect.

Action for a Charged Particle in an Electromagnetic Field

Of course, like Newton's Laws and Maxwell's equations, our result here is ultimately derived from experimental observation. However, it turns out to have a surprising simplicity in the 4-potential $\left(\varphi, \vec{A}\right)$ formalism.

We found, using the nonrelativistic limit to fix the overall constant, that for a free particle the only option for the action was

$$S = -mc^2 \int_a^b d\tau,$$

and hence Lagrangian

$$L = -mc^2 \sqrt{1 - \frac{v^2}{c^2}}.$$

The momentum

$$\vec{p} = \frac{\partial L}{\partial \vec{v}} = \frac{m\vec{v}}{\sqrt{1 - v^2 / c^2}}.$$

The energy is defined as

$$E = \vec{p} \cdot \vec{v} - L = \frac{mc^2}{\sqrt{1 - v^2 / c^2}}.$$

The principle of least action is that along the physical path

$$\delta S=-mc\int_a^b d\tau=0,$$

led to the unsurprising result of the physical path between two points in space time being one at constant velocity, that being the path of maximum proper time.
In an electromagnetic field, we can only add to the action Lorentz invariant terms in the integral along the path. Since the fields are defined by the four potential $A^\mu=\left(\varphi/c,\vec{A}\right)$, the simplest possible action for a single charged particle is

$$S=\int_a^b\left(-mc^2d\tau-eA_\mu dx^\mu\right).$$

Writing the action in terms of the Lagrangian

$$S=\int_a^b Ldt,$$

we find

$$L=-mc^2\sqrt{1-v^2/c^2}+e\vec{v}\cdot\vec{A}\left(\vec{r},t\right)-e\varphi\left(\vec{r},t\right).$$

This does indeed lead to the Lorentz force law, $\vec{F}=e\left(\vec{E}+\vec{v}\times\vec{B}\right)$, see my notes here.
Recalling now the classical mechanics definition of momentum:

$$\vec{p}=\frac{\partial L}{\partial\vec{v}}=\frac{m\vec{v}}{\sqrt{1-v^2/c^2}}+e\vec{A}\left(\vec{r},t\right),$$

(this is called the *canonical* momentum) and energy

$$E=\vec{p}\cdot\vec{v}-L=\frac{mc^2}{\sqrt{1-v^2/c^2}}+e\varphi\left(\vec{r},t\right).$$

So the energy is just the mass energy (rest plus kinetic) plus the electrostatic potential energy, the magnetic field does no work on the particle, so doesn't appear, as usual. This is just the Hamiltonian, having velocity and position as variables, as opposed to the Lagrangian, which is written in terms of momentum and position. *But* if you want to use the Hamiltonian to derive Hamilton's equations of motion, it must be written in terms of the canonical momenta, and the vector potential reappears!
The two equations can be written for the canonical four momentum $P^\mu=\left(E/c,\ \vec{p}\right)$ as

$$P^\mu=mU^\mu+eA^\mu.$$

It can also be shown that the equation of motion is

$$m\frac{d^2x^\mu}{d\tau^2}=eF^{\mu\nu}\frac{dx^\nu}{d\tau},$$

or

$$m\frac{dU^\mu}{d\tau}=eF^{\mu\nu}U_\nu.$$

Exercise: check this out for some simple cases.

25. Motion of a Charged Particle in a Slowly Varying Magnetic Field

Following Jackson 12.5
The circling motion of a particle in a constant magnetic field is well known, the general motion includes a constant drift velocity in the direction of the field. If the field is slowly varying, the motion is still the drifting circle, but now the center, called in this context the guiding center, follows a nontrivial path. If the motion of the guiding center is much slower than the circling motion, the adiabatic theorem is valid. Recall that for a periodic motion the action integral is $J=\oint pdq$ around one cycle, and if the external parameters vary sufficiently slowly, this integral stays constant.

Surprisingly, perhaps, this is easiest to understand in quantum mechanics: imagine a particle trapped in a tube, with a momentum corresponding to some wavelength so the tube is an integral number of wavelengths long, $n\lambda=L$. This means that, using $p=h/\lambda$, we have $pL/h=n$, the integer n being the number of times the phase winds around on going once around the tube. Now suppose you gradually introduce some potential in the tube, so the particle is now on a roller coaster. The wavelength will now be different in different places, so there will be a local value of p. However, if this is done smoothly, the wave function will distort smoothly, and it will still wind around the same number of times. This is a traveling wave, the wave function won't go to zero anywhere. Nowadays we would say the integer n is topologically protected. Anyway, this is the action integral.

The simple classical dynamics example is a swinging pendulum whose length is gradually change, a string going over a pulley before hanging down. It turns out that the energy of motion of the pendulum increases in direct proportion as the frequency does. This is easy to understand quantum mechanically: if a simple harmonic oscillator is in the n^{th} quantum excited state, its wave function has n zeroes, and if the potential is gradually changed in strength (but not shape) that state will smoothly change, but keep the same number of zeroes, so the energy will track the frequency.

The corresponding situation here is the gradual drift of the guiding center into a magnetic field of different strength.

The canonical momentum is

$$P_i = \frac{\partial L}{\partial u_i} = \gamma m u_i + eA_i.$$

The action corresponding to the circling motion perpendicular to the field is

$$J = \oint \vec{P}_\perp \cdot d\vec{\ell} = \oint \gamma m \vec{v}_\perp \cdot d\vec{\ell} + e\oint \vec{A}\cdot d\vec{\ell} \ .$$

In the adiabatic limit, we can take $\vec{v}_\perp \parallel d\vec{\ell}$ so (with $v_\perp = a\omega_B,\ d\ell = ad\theta$)

$$\begin{aligned} J &= \oint \gamma m\omega_B a^2 d\theta + e\oint \vec{A}\cdot d\vec{\ell}, \\ &= 2\pi\gamma m\omega_B a^2 + e\int_S \vec{B}\cdot\hat{\vec{n}}da. \end{aligned}$$

At this point, it's crucial to get the signs right, because these terms partially cancel.
Let's suppose the magnetic field is pointing in the positive z direction, and the circling motion is therefore in the x, y plane. Take the particle to be positively charged, and think of the moment it crosses the positive x axis, as it circles the origin. The force $e\vec{v}\times\vec{B}$ has to be towards the center of its circle, the origin, so it must be moving downwards, meaning circling clockwise. Since it's part of the same integral, the $\oint \vec{A}\cdot d\vec{\ell}$ must also be taken clockwise, so that second integral must give

$$e\int_S \vec{B}\cdot\hat{\vec{n}}da = -e\pi a^2 B.$$

Noting that $\omega_B = eB/\gamma m,$ we find the first term in J is twice the magnitude of the

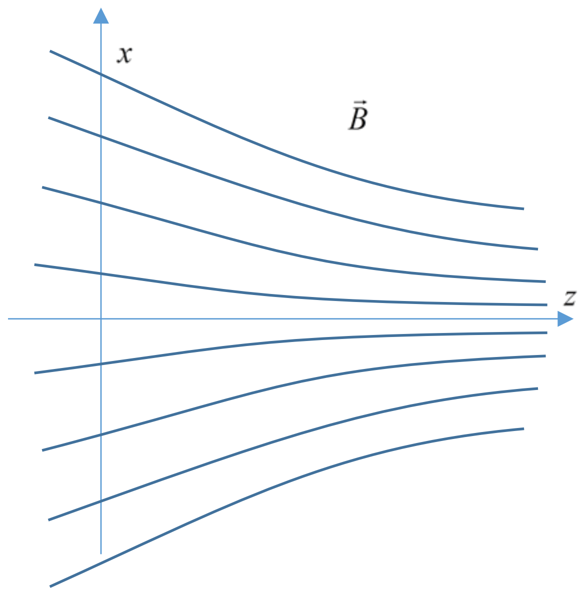
second, and positive, so

$$J = e\left(B\pi a^2\right).$$

This means that as the guiding center moves from a weak field to a strong field, the circle orbit will shrink to keep *the same total magnetic flux through the circle*.

An equivalent invariant is $p_\perp^2 / B$. (*Exercise*: prove it!)

This invariant gives some insight into why a circling charged particle can be repelled by a magnetic pole, such as one of the Earth's magnetic poles. Jackson considers the case of a field mainly in the z direction, but of varying strength:



Imagine a charged particle, circling in the x, y plane with velocity $\vec{v}_\perp$, but also coming in from the left with a nonzero drift velocity $v_\parallel$ in the z direction.

Since the only force on the particle is from the static magnetic field, its kinetic energy cannot change,

$$v_\perp^2 + v_\parallel^2 = v_0^2.$$

From the adiabatic theorem,

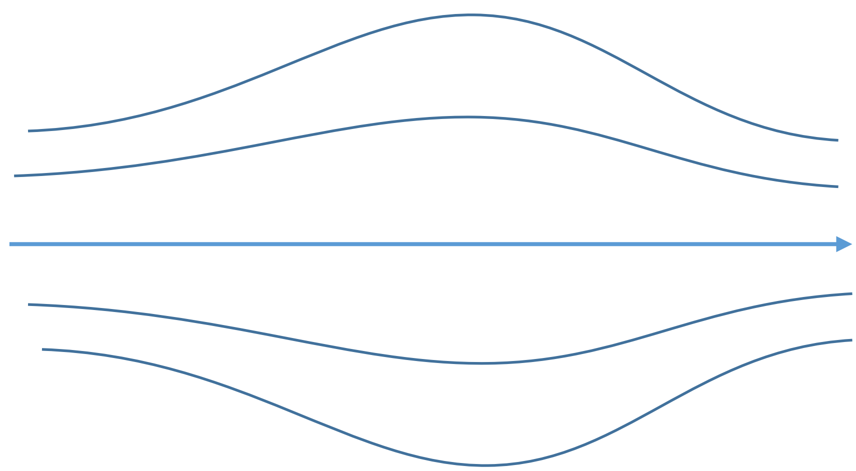
$$\frac{v_\perp^2}{B} = \frac{v_{\perp 0}^2}{B_0},$$

so

$$v_\parallel^2 = v_0^2 - v_{\perp 0}^2 \frac{B(z)}{B_0}.$$

This implies that motion in the z direction is the same as a particle in a one-dimensional potential, evidently with a turning point if the field strength gets sufficiently strong to make the right hand side zero.

A charged particle can be confined in a "magnetic bottle" having field lines coming together at both ends. This is the basic idea behind the tokomak and other attempts to confine a plasma at very high temperatures in hopes of gaining confined nuclear fusion. Unfortunately, the one-particle model ignores the plasma interactions occurring at finite density, and these render the confinement unstable.



However, there are excellent examples in nature of this kind of magnetic bottle: the van Allen radiation belts toroidally shaped region surrounding the Earth, and filled with charged particles.

These are shaped by the Earth's magnetic field: